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MATHEMATICAL PHYSICS

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Abstract

Full Text

MATHEMATICAL PHYSICS

V. N. ZHIGULEV

ON THE THEORY OF ORDERED STATISTICAL SYSTEMS

(Presented by Academician N. N. Bogolyubov, 25 VII 1964)

A method is proposed for solving the Bogolyubov equations for correlation distribution functions in the case of a small van der Waals number for an inhomogeneous gas, which makes it possible not to assume molecular chaos. A chain of equations is obtained for an ideal gas, a particular solution of which is the solution under the assumption of molecular chaos, leading to the Boltzmann equation for the one-particle distribution function; a solution of the chain of equations for an ideal gas is also obtained in the weakly ordered case, reducing to a system of two integro-differential equations; a general investigation of this system is carried out, including the derivation of hydrodynamic equations in the case of small Knudsen number; it is shown that these flows possess, in a certain sense, the property of conservation of correlations, leading to the appearance of 15 new conservative quantities analogous to density, temperature, and mean velocity; the one-particle distribution function in the equilibrium weakly ordered case is no longer a Maxwell function.

1. The Bogolyubov equations for s-particle correlation functions have the form (see (1), the notation is the same)

$$\frac{\partial F^{(s)}}{\partial t} = [H_s F^{(s)}] + \frac{1}{v} \int \sum_{1 \le i \le s} [\Phi_{i,s+1} F^{(s+1)}] \, dx_{s+1} \quad (s = 1, 2, ...). \tag{1}$$

We shall consider a monatomic gas with short-range forces of intermolecular interaction (acting over lengths of order r_0); the Knudsen number $K = l/L$ (the mean free path $l = v/r_0^2$, L is the characteristic linear scale of the flow) is of order unity or smaller; the characteristic time scale $t_0 \sim L/u_0$, where u_0 is the characteristic magnitude of the molecular velocity; the Mach number is of order unity; the van der Waals number $\varepsilon = r_0^3/v$ is assumed small.

In accordance with the assumptions made concerning the metric case under consideration, the function $F^{(s)}$ has two characteristic lengths of variation: if the relative configuration of s particles does not change, then $F^{(s)}$ changes on the scale l; otherwise the scale of variation of $F^{(s)}$ will be r_0.

For the metric case under consideration, an estimate of the orders of magnitude of the terms entering the system of equations (1) gives: a) in the equation for $F^{(1)}$, all terms have the same order of magnitude; b) in the equations for $F^{(s)}$, if $|\mathbf{q}_i - \mathbf{q}_j| \sim r_0$ (i, j are any two points among s), there are large terms of order $1/\varepsilon$ in comparison with the others; but if all $|\mathbf{q}_i - \mathbf{q}_j| \sim l$, then some of these terms become small in comparison with the others. Therefore the system (1) may be solved by the method of successive approximations, where the n-th approximation has the form:

1) $|\mathbf{q}_i - \mathbf{q}_j| \sim l$;

$$\frac{d_s F_n^{(s)}}{dt} = \frac{1}{v} \int \sum_{1 \leq i \leq s} [\Phi_{i,s+1} F_n^{(s+1)}] \, dx_{s+1}; \tag{2}$$

2) $|\mathbf{q}_i - \mathbf{q}_j| \sim r_0$:

$$\sum_{1 \leqslant i \leqslant s} (c_{s+1} - c_i) \frac{\partial F_n^{s+1}}{\partial \vec{\xi}_i} =$$

$$= \sum_{1 \leqslant i < j \leqslant s+1} [\Phi_{ij} F_n^{(s+1)}] - \frac{d F_{n-1}^{(s+1)}}{dt} + \frac{1}{v} \int \sum_{1 \leqslant i \leqslant s+1} [\Phi_{i,s+2} F_{n-1}^{(s+2)}] \, dx_{s+2}, \tag{3}$$

$$\vec{\xi}_i = \mathbf{q}_{s+1} - \mathbf{q}_i, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + c_1 \frac{\partial}{\partial \mathbf{r}}, \qquad \mathbf{r} = \mathbf{q}_i, \qquad \frac{d_s}{dt} = \frac{\partial}{\partial t} + \sum_{1 \leqslant i \leqslant s} c_i \frac{\partial}{\partial \mathbf{q}_i}.$$

The n-th approximation is meaningful for sufficiently short-range forces of intermolecular interaction; thus, if $\Phi_{ij} = m n_0^2 (r_0 / |\mathbf{q}_i - \mathbf{q}_j|)^\alpha$, then α must be greater than n.

Characteristic for the case under consideration is the splitting of system (1) into the system of equations (2) in macrospace ($\sim l$) and the system (3) in microspace ($\sim r_0$).

2. In the first approximation the system of equations (3) is greatly simplified, and we can exclude the microspace ($\sim r_0$) from further consideration. From the point of view of macrospace, s points are separated on the scale r_0 to infinity and, consequently, the integral in system (2) is the sum of all possible binary interactions; one integration can be carried out; as a result, substituting (3) into (2), we obtain a single approximate (in the sense of the leading term in ε) system of equations for determining the correlation functions "before collision" in macrospace for an ideal gas:

$$\frac{d_{sF}^{(s)}}{dt} = \frac{1}{v} \sum_{1 \leqslant i \leqslant s} \int \{F^{(s+1)}\}_i \, dP_{i,s+1} \qquad (s = 1, 2, \ldots),$$

$$\{F^{(s+1)}\}_i = F^{(s+1)}(t, x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_s, x'_{s+1}) - F^{(s+1)}(t, x_1, \dots, x_{s+1}), \tag{4}$$

$$dP_{i,s+1} = b\,db\,d\varepsilon'\,|c_i - c_{s+1}|\,dp_{s+1};$$

b is the impact parameter; ε' is the azimuthal angle; x_i is the aggregate $c_i\mathbf{q}_i$; the quantities x'_i, x'_{s+1}, x_i, x_{s+1} have the same coordinate values in physical space, equal to $\mathbf{q}_i$; c'_i, c'_{s+1} are related to c_i, c_{s+1} by the law of binary collision (2).

The functions $F^{(s)}$ $(s > 1)$ in macrospace $(\sim l)$ are not single-valued; therefore, for definiteness, we consider the branch corresponding to the value of the function $F^{(s)}$ before all possible binary interactions (“before collision”).

3. Let us turn to the analysis of the chain of equations (4) for an ideal gas. In deriving the usual Boltzmann kinetic equation one uses the hypothesis of molecular chaos, which consists in assuming approximately everywhere in macrospace that

$$F^{(2)}(t, \mathbf{r}, c_1, c_2) = F^{(1)}(t, \mathbf{r}, c_1) F^{(1)}(t, \mathbf{r}, c_2).$$

Theorem 1. *The chain of equations (4) for an ideal gas has the particular solution*

$$F^{(s)}(tx_1, x_2 \dots, x_s) = \prod_{1 \leqslant i \leqslant s} F^{(1)}(t, x_i); \tag{5}$$

the function $F^{(1)}(t, \mathbf{r}, c_1)$ *then satisfies the Boltzmann equation*

$$\frac{dF^{(1)}}{dt} = \frac{1}{v} \int \{F^{(1)}(t, \mathbf{r}, c_1) F^{(1)}(t, \mathbf{r}, c_2)\}_1 \, dP_{12}. \tag{6}$$

The theorem is established by substituting the solution (5) into system (4). The theorem is directly generalized to the next approximation (real gas).

Thus, if at some instant t_0 the system satisfies the property of molecular chaos (5), then, in the presence of stability of the solution and of boundary conditions compatible with property (5), the correlation functions $F^{(s)}$ will always be expressed by the solution (5); in other words, the property of molecular chaos (disorder) is a conserved property of the gas. (Cf. the works (^{3},^{4}), where an analogous result was obtained in the hydrodynamic case ($K \ll 1$).)

The natural question arises of the formation of the property of molecular chaos: will not all solutions for $F^{(s)}$ tend to the solution (5)? In the subsequent sections a negative answer is given to this question.

4. Consider the case when $\omega = g(t, x_1, x_2)/F^{(1)}(t, x_1)F^{(1)}(t, x_2)$ is a small quantity $(g(t, x_1, x_2) = F^{(2)}(t, x_1, x_2) - F^{(1)}(t, x_1)F^{(1)}(t, x_2))$.

Theorem 2. In the case $|\omega| \ll 1$, the system of equations (4) has the following solution for the functions $F^{(s)}$ $(s \geqslant 2)$:

$$F^{(s)} = \prod_{1 \leqslant i \leqslant s} F^{(1)}(t, x_i) + \sum_{1 \leqslant i < j \leqslant s} g(t, x_i, x_j) \prod_{\substack{1 \leqslant k \leqslant s \\ k \neq i, j}} F^{(1)}(t, x_k); \tag{7}$$

the functions $F^{(1)}$ and g satisfy the system of integro-differential equations

$$\frac{d_1 F^{(1)}}{dt} = \frac{1}{v} \int \{F^{(1)}(x_1)F^{(1)}(x_2)\}_1 \, dP_{12} + \frac{1}{v} \int \{g(x_1, x_2)\}_1 \, dP_{12},$$

$$\frac{d_2 g}{dt} = \frac{1}{v} \int \{F^{(1)}(x_i)g(x_3, x_2) + F^{(1)}(x_3)g(x_1, x_2)\}_1 \, dP_{13} +$$

$$+\frac{1}{v} \int \{F^{(1)}(x_2)g(x_1, x_3) + F^{(1)}(x_3)g(x_1, x_2)\}_2 \, dP_{23}. \tag{8}$$

The theorem is proved by substituting expression (7) into the system of equations (4) and neglecting quantities of order ω^2.

It follows from Theorem 2 that if the initial and boundary conditions for $F^{(s)}$ do not contradict expressions (7) and the solution is stable, then the solution (7) holds everywhere—the weakly ordered case of the flow of an ideal gas, which is considered below.

5. In the homogeneous case $(\partial/\partial q_i = 0, \ i = 1, 2)$ of a weakly ordered ideal gas, the following holds.

Theorem 3. If $\partial/\partial q_i = 0$ $(i = 1, 2)$ and $\{F^{(1)}(c_1)F^{(1)}(c_2)\} \sim \omega$, then the quantity $H^* = 2\int g^{\omega} \, dc_1 dc_2$ always decreases, tending to a stationary $(d/dt = 0)$ value.

Indeed, for the derivative $dH^*(t)/dt$ the following expression can be obtained (if one uses the integral theorem (5,28), Chap. III (^{2}), as well as the condition of Theorem 3 that $|F^{(1)}(c_1')F^{(1)}(c_2') - F^{(1)}(c_1)F^{(2)}(c_2)| \sim \omega$):

$$\frac{dH^*}{dt} = -\frac{1}{2v} \int F^{(1)}(c_1)F^{(1)}(c_2)F^{(1)}(c_3)[(\omega(c_1', c_2) + \omega(c_3', c_2)$$

$$-\omega(c_1, c_2) - \omega(c_3, c_2))^2 \, dP_{13} + (\omega(c_1, c_2') + \omega(c_1, c_3)$$

$$-\omega(c_1, c_2) - \omega(c_1, c_3))^2 \, dP_{23}] \, dc_1 dc_2 + O(\omega^3). \tag{9}$$

It follows from Theorem 3 that a homogeneous system tends to statistical equilibrium, since for the principal part $F^{(1)}(t, c_2)$, differing from its exact value by a quantity of order ω, Boltzmann' s H-theorem is valid and, consequently, the system arrives at a state where the basic condition of Theorem 3 is valid.

In statistical equilibrium the equations are satisfied:

$$\omega(\mathbf{c}_1', \mathbf{c}_2) + \omega(\mathbf{c}_3', \mathbf{c}_2) = \omega(\mathbf{c}_1, \mathbf{c}_2) + \omega(\mathbf{c}_3, \mathbf{c}_2),$$

$$\omega(\mathbf{c}_1, \mathbf{c}_2') + \omega(\mathbf{c}_1, \mathbf{c}_3') = \omega(\mathbf{c}_1, \mathbf{c}_2) + \omega(\mathbf{c}_1, \mathbf{c}_3), \tag{10}$$

which indicates that the function $\omega(\mathbf{c}_1, \mathbf{c}_2)$ in the equilibrium case is the following combination of invariants of binary collision:

$$\begin{aligned}\omega(\mathbf{c}_1, \mathbf{c}_2) = a_1 + \vec{a}_2(m\mathbf{c}_1 + m\mathbf{c}_2) + \tilde{a}_3(m\mathbf{c}_1 m\mathbf{c}_2 + m\mathbf{c}_2 m\mathbf{c}_1) + \\ + a_4(mc_1^2/2 + mc_2^2/2) + \\ + \vec{a}_5(m\mathbf{c}_1 mc_2^2/2 + m\mathbf{c}_2 mc_1^2/2) + a_6(mc_1^2/2)(mc_2^2/2).\end{aligned} \tag{11}$$

The quantities a_i are constants, of which a_1, a_4, a_6 are scalars, $\vec{a}_2$ and $\vec{a}_5$ are vectors, $\tilde{a}_3$ is a symmetric tensor of rank two—a total of 15 scalar quantities.

6. Theorem 4. A consequence of the equation for the function g is 15 transport equations.

Theorem 5. In the case of a small Knudsen number ($K \ll 1$)

$$F^{(1)} = F_0^{(1)} + KF_1^{(1)} + KF_2^{(1)} + \dots, \qquad g = g_0 + Kg_1 + K^2 g_2 + \dots. \tag{12}$$

The necessary solvability conditions for the system of recurrence equations for the functions $F_j^{(1)}$ and g_j yield a successively refined hydrodynamic system of equations, which is a closed system of 20 equations with respect to 20 scalar functions (of t and $\mathbf{r}$)—n, T, $\mathbf{V}$, a_i.

The 15 transport equations for the quantities a_i are obtained from the equation for the function g by multiplying it by 1, $m\mathbf{c}_1 + m\mathbf{c}_2$, $mc_1^2/2 + mc_2^2/2$, $m\mathbf{c}_1 m\mathbf{c}_2$, ..., and then integrating over the spaces $\mathbf{c}_1$ and $\mathbf{c}_2$. The integrals of the collision terms then vanish by virtue of the fundamental property of the invariants.

Theorem 4 indicates the existence, for the weakly ordered case under consideration of the flow of an ideal gas, of 15 new conservative quantities, for which there are 15 additional transport equations. This indicates that the correlations possess, in a certain sense, a conservation property.

It is of great interest to consider the indicated hydrodynamic equations, since equations (8) are an apparatus of the theory of turbulence formulated in terms of the aerodynamics of a rarefied gas.

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Note: Figure translations are in progress. See original paper for figures.

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