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Abstract

Full Text

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ON SOME MODIFICATIONS OF THE CONCEPT OF THE MODULUS OF SMOOTHNESS AND THEIR APPLICATIONS

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1°. We shall use the following notation. A function $f(x) \in L_{2\pi}$; $\omega_k(\delta, f)_{L^p}$ is its modulus of smoothness of order k in the metric L^p ($L^\infty = C_{2\pi}$); $E_n(f)_{L^p}$ denotes best approximations by trigonometric polynomials of degree not exceeding n; $\sigma(f)$ is the Fourier series; $c > 0$ is a finite constant depending only on those arguments which will be written out. Put

$$ {}^s\Delta_t^r f(x) = \sum_{k=0}^{r} (-1)^k C_r^k f[x + (r-2k)t], $$

$$ \sigma(f) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx), $$

$$ F(x) = \sum_{k=1}^{\infty} \left(-\frac{b_k}{k} \cos kx + \frac{a_k}{k} \sin kx \right). $$

In a number of questions it is possible to characterize the structural properties of a function by the moduli of smoothness of the primitive (and not of the function itself!), and also by the quantities introduced below (the L-modulus of smoothness). This makes it possible to refine some well-known results.

2°. Absolute convergence of Fourier series. Numerous works are devoted to the question of absolute convergence. The fundamental theorems are those of S. N. Bernstein ([1]), pp. 217-223, 166-169. A detailed exposition of the results pertaining to this subject is given in the book of N. K. Bari ([2]).

Theorem 1. Let the sequence $n_k \uparrow \infty$ be such that

$$ \sum_{z=k}^{\infty} \frac{(a_{n_z}^2 + b_{n_z}^2)}{z^2} \leq C_1 \sum_{n=n_k}^{\infty} \frac{(a_n^2 + b_n^2)}{n^2}. \tag{1} $$

Then

$$\sum_{k=1}^{\infty}(|a_{n_k}|+|b_{n_k}|) \leq \frac{2\sqrt{2C_1}}{\sqrt{\pi}} \sum_{k=1}^{\infty} \sqrt{k}\, E_{n_k-1}(F)_{L^2}.$$

Corollary 1. Let $f(x) \in L_{2\pi}$. Then

$$\sum_{k=1}^{\infty}(|a_k|+|b_k|) \leq \frac{2\sqrt{2}}{\sqrt{\pi}} \sum_{k=1}^{\infty} \sqrt{k}\, E_{k-1}(F)_{L^2}.$$

Corollary 2. If $E_n(f)_L = O(1/n)$, then

$$\sum_{k=1}^{\infty}(|a_k|+|b_k|) \leq C_2(r) \sum_{k=1}^{\infty} k^{-1/2} \left[\omega_k\left(\frac{1}{k}, F\right)_C\right]^{1/2}.$$

There exist absolutely continuous functions satisfying the condition of Corollary 2 and not satisfying Zygmund' s condition.

([2], p. 614)

$$\sum_{k=1}^{\infty} k^{-1} \sqrt{\omega_1(1/k, f)_C} < +\infty$$

and the stronger condition

$$\sum_{k=1}^{\infty} k^{-1} \sqrt{\omega_2(1/k, f)_C} < +\infty.$$

Theorem 2. Let the sequence $n_k \uparrow \infty$ satisfy condition (1), and let $\Phi(u)$ be such that $\Phi(0)=0$, $\Phi(u)$ is increasing, $\Phi(u)$ is convex upward, and

$$\Phi(u_1 \times u_2) \leq \Phi(u_1)\Phi(u_2).$$

Then

$$\sum_{k=1}^{\infty} \Phi\left(a_{n_k}^2 + b_{n_k}^2\right) \leq \Phi(4)\,\Phi\left(\frac{C_1}{\pi} E_{n_{k-1}}^2(F)_{L^2}\right) +$$

$$+2\sum_{k=1}^{\infty} \Phi(8k)\Phi\left(\frac{C_1}{\pi} \times E_{n_{k-1}}^2(F)_{L^2}\right).$$

Definition 1. The quantity

$$\widetilde{L}_p^r(h,f) = \frac{1}{h} \sup_{0 \le u \le h} \left\| \int_0^u {}^s\Delta_t^r f(x)\,dt \right\|_{L^p}$$

will be called the $\widetilde{L}$-modulus of smoothness of order r of the function $f(x)$ in the metric L^p.

Let us note a number of properties of the $\widetilde{L}$-modulus of smoothness:

1. If, as $h \to 0$, $\widetilde{L}_1^{(r)}(h,f) = o(h^2)$, then $f(x) \sim \text{const}$.
2. In order that $f(x) \in L^p$, where $1 \le p \le \infty$, have an $(r-1)$-st derivative belonging to $\text{Lip}(1,p)$, it is necessary and sufficient that

$$L_p^{(r)}(h,f) = O(h^2).$$

3. Let $f(x) \in L^2$. In order that

$$\omega_r(1/n,f)_{L^2} \sim \widetilde{L}_2^{(r)}(1/n,f), \tag{2}$$

it is necessary and sufficient that

$$E_n|f|_{L^2} \le C_3(r)\widetilde{L}_2^{(r)}(1/n,f).$$

Remark. Equality (2) is not always fulfilled.

Theorem 3. Let $f(x) \in L$ (r odd) and $f(x) \in L^2$ (r even). If $0 < m < 2$, then

$$\sum_{k=n}^{\infty} (|a_k|^m + |b_k|^m) \le C_4(r,m) \left\{ \sum_{k=n}^{\infty} k^{-m/2} \left[\widetilde{L}_2^{(r)}\left(\frac{1}{k}, f\right) \right]^m + \left[\widetilde{L}_2^{(r)}\left(\frac{1}{n}, f\right) \right] n^{1-m/2} \right\}.$$

Theorem 4. Let $f(x) \in L$.

1. If $a_k \ge 0$ $(k = 1, 2, ...)$, then* for r odd

$$n^{-r} \sum_{k=1}^{n} a_k k^{r-1} \le C_5(r) \widetilde{L}_C^{(r)}\left(\frac{1}{n}, F\right).$$

2. If $b_k \ge 0$ $(k = 1, 2, ...)$, then for r even

$$n^{-r} \sum_{k=1}^{n} b_k k^{r-1} \le C_6(r) \widetilde{L}_C^{(r)}\left(\frac{1}{n}, F\right).$$

3. If $a_k \ge 0$ $(k = 1, 2, ...)$, then

$$\sum_{k=n}^{\infty} a_k \leq C_7 \sum_{k=n}^{\infty} \omega_r\left(\frac{1}{k}, F\right)_C.$$

4. If $b_k \geq 0$ $(k = 1, 2, \ldots)$, then

$$\sum_{k=n}^{\infty} b_k \leq C_8 \sum_{k=n}^{\infty} \omega_r\left(\frac{1}{k}, F\right)_C.$$

* No restrictions are imposed on b_k.

3°. **Multipliers of uniform convergence.** (A survey of known results is given in the work of F. I. Kharshiladze ([3]).) Let

$$\Lambda_n(t) = \lambda_0/2 + \sum_{k=1}^{n} \lambda_k \cos kt.$$

Theorem 5. *If the conditions*

$$\int_0^{2\pi} \left|\sum_{\nu=0}^{n} \Lambda_\nu(t)\right| dt = O(n), \qquad E_{[(n+1)/2]}(F)_C \int_0^{2\pi} |\Lambda_n(t)| dt = O\left(\frac{1}{n}\right),$$

are satisfied, then for a continuous $f(x)$ the numbers λ_k are multipliers of uniform convergence.

Remark. Already for $\lambda_k \equiv 1$ $(k = 0, 1, 2, \ldots)$ there exist functions satisfying the conditions of Theorem 5, whereas the conditions of the theorems of Bojanic ([4]) and Kharshiladze ([3]) are not fulfilled. An example may be

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos 2^n x}{n^{1+\alpha}} \quad (0 < \alpha < 1) \quad ((^2),\ \text{pp. } 294\text{–}296).$$

4°. The concept of the $\widetilde{L}$-modulus of smoothness has applications in the theory of singular integrals. Relying on the properties of the $\widetilde{L}$-modulus, one can prove the following theorems.

Theorem 6. *The following relations are equivalent:*

$$f(x) \in \mathrm{Lip}(1, p),$$

$$\left\| \frac{3}{2\pi n(2n^2+1)} \int_{-\pi}^{\pi} |f(x+t) - f(x)| \left(\frac{\sin nt/2}{\sin t/2}\right)^4 dt \right\|_{L^p} = O\left(\frac{1}{n}\right),$$

$$\left\| \frac{(2n)!!}{(2n+1)!!} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x+t) - f(x)| \cos^{2n} \frac{t}{2} \, dt \right\|_{L^p} = O\left(\frac{1}{\sqrt{n}}\right).$$

Theorem 7. *In order that the relation*

$$\left\| \frac{(2n)!!}{(2n-1)!!} \int_{0}^{\pi} |f(x+t) - 2f(x) + f(x-t)| \cos^{2n} \frac{t}{2} \, dt \right\|_{L^p} = O\left(\frac{1}{\sqrt{n^\alpha}}\right),$$

hold, where $0 < \alpha \leq 2$, *it is necessary and sufficient that, as* $t \to 0$,

$$\omega_2(t, f) = O(t^\alpha).$$

Theorem 8. *In order that, as* $r \to 1 - 0$,

$$\left\| \frac{1}{2\pi} \int_{0}^{\pi} |f(x+t) - 2f(x) + f(x-t)| \frac{1 - r^2}{1 - 2r \cos t + r^2} \, dt \right\|_{L^p} = O((1-r)^\alpha),$$

where $0 < \alpha < 1$, *it is necessary and sufficient that* $f(x) \in \mathrm{Lip}(\alpha, p)$.

5°. **Generalized methods of summation of Fourier series in a Hilbert space.** Let $H = \{x\}$ be a Hilbert space, $\{x_n\}$ $(n = 1, 2, ...)$ a complete orthonormal system in H,

$$E_n(x)_H = \left[\sum_{k=n+1}^{\infty} (x, x_k)^2 \right]^{1/2};$$

to each element $x \in H$ there is assigned the series

$$U(x, \xi) = \sum_{k=1}^{\infty} \gamma_k(\xi)(x, x_k) x_k,$$

where the functions $\gamma_k(\xi)$ are given on some set with a limit point w. We assume that $\gamma_1(\xi) = 1$ and

$$\sum_{k=1}^{\infty} \gamma_k^2(\xi)(x, x_k)^2 < +\infty,$$

at least for values ⋯

functions ξ close to w. In what follows, the summability method is considered only for such ξ. M. F. Timan (([5]), p. 351; ([6])) proved that

$$\omega_r\left(\frac{1}{n}, f\right)_{L^2} \sim \frac{1}{n^r}\left[\sum_{k=1}^{n} k^{2r-1} E_k^2(f)_{L^2}\right]^{1/2}.$$

Definition 2. The quantity

$$\tilde{\omega}_r\left(\frac{1}{n}, x\right)_H = \frac{1}{n^r}\left[\sum_{k=1}^{n} k^{2r-1} E_k^2(x)_H\right]^{1/2}$$

will be called the $\tilde{\omega}$-modulus of smoothness of order r of the element x in the space H.

Definition 3. Let $\varphi(\xi)$ be such that $\lim_{\xi\to 0}\varphi(\xi) = 0$, and for all $x \in H$ ($x \in \{Cx_1\}$),

$$\|x - U(x,\xi)\|_H \geq C_9(x)\varphi(\xi)$$

and there exists $x \in H$ ($x \in \{Cx_1\}$) for which

$$\|x - U(x,\xi)\|_H \leq C_{10}(x)\varphi(\xi). \tag{3}$$

Then we shall say that the summability method is saturated in H with order $\varphi(\xi)$, and call the class of saturation the set of $x \in H$ for which (3) is fulfilled.

Theorem 9. Let $x \in H$, $r > 0$. Then the series

$$\sum_{k=1}^{\infty} k^{2r-1} E_k^2(x)_H, \qquad \sum_{k=1}^{\infty} k^{2r}(x, x_k)^2, \qquad \sum_{k=1}^{\infty} k^{2r-1}\tilde{\omega}_{r+1}^2\left(\frac{1}{k}, x\right)_H$$

converge or diverge simultaneously.

Remark 1. Let $f(x) \in L^2_{2\pi}$. Then the series $\sum_{k=1}^{\infty}(a_k^2 + b_k^2)k^{2r}$ $(r \geq 1)$ and the integral

$$\int_0^{2\pi}\int_0^{2\pi} \frac{\left|{}^{s}\Delta_t^{r+1} f(x)\right|^2}{t^{2r+1}}\,dt\,dx$$

converge or diverge simultaneously.

Remark 2. For $r = 0$, the theorem and Remark 1 do not hold (([2]), pp. 347, 399).

Theorem 10. If $x \in H$, then

$$\|x - U(x,\xi)\|_H = \left[\sum_{k=1}^{\infty} E_k^2(x)_H \left[(1-\gamma_{k+1}(\xi))^2 - (1-\gamma_k(\xi))^2\right]\right]^{1/2}.$$

Corollary. Let $\gamma_k(n) = 1 - ((k-1)/n)^p$, if $k = 1, \dots, n$, and $\gamma_k(n) = 0$ for $k > n$, where $p > 0$ is any real number. Then

$$\frac{1}{n^p}\left[E_1^2(x)_H + 2p\sum_{k=2}^{n}(k-1)^{2p-1}E_k^2(x)_H\right]^{1/2} \le$$

$$\le \|x - U^{(p)}(x,n)\|_H \le \frac{\sqrt{2p}}{n^p}\left[\sum_{k=1}^{n} k^{2p-1}E_k^2(x)_H\right]^{1/2}.$$

The summability method is saturated with order $1/n^p$. The class of saturation is the set of elements for which

$$\sum_{k=1}^{\infty} k^{2p-1}E_k^2(x)_H < +\infty.$$

Relying on Theorem 10, it is not difficult to obtain analogous results for other summability methods as well.

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Note: Figure translations are in progress. See original paper for figures.