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Abstract

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ESTIMATE OF THE COEFFICIENTS OF UNIVALENT FUNCTIONS

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Let, as usual, S be the class of functions $f(z) = z + c_2 z^2 + \cdots$, regular and univalent in the disk $|z| < 1$, and let Σ be the class of functions $F(z) = z + \alpha_0 + \alpha_1/z + \cdots$, regular and univalent in the domain $1 < |z| < \infty$.

Each function $F(z) \in \Sigma$ generates a system of functions $\{A_n(z)\}$ $(n = 1, 2, \ldots)$ by means of the expansion

$$\ln \frac{z - t}{F(z) - F(t)} = \sum_{n=1}^{\infty} A_n(t) z^{-n}, \qquad |z| > 1, \quad |t| > 1 \tag{1}$$

(the branch of the logarithmic function is taken which tends to zero as $z = \infty$). In the author' s paper ([1]) the basic properties of the system $\{A_n(z)\}$ are set forth and, in particular, for every z in the domain $|z| > 1$ the sharp inequality is established:

$$\sum_{n=1}^{\infty} n |A_n(z)|^2 \leq \ln \frac{1}{1 - r^2}, \qquad |z| = \rho = \frac{1}{r} > 1. \tag{2}$$

Using this inequality, it is possible to improve the estimate of the coefficients of univalent functions of the class S. Below we give the derivation of a new estimate.

Theorem 1. For a function

$$f(z) = z + \sum_{n=2}^{\infty} c_n z^n \in S$$

and for every $\rho \in [1, \infty)$, the inequality

$$|c_n| \leq \rho^{n-1} \max_{|t| = \rho} \left| \left\{ \exp \left[\sum_{k=1}^{n-1} A_k(t) z^{-k} \right] \right\}_{z=t}^{(n-1)} \right|, \qquad n = 2, 3, \ldots, \tag{3}$$

holds, where the system $\{A_k(z)\}$ $(k = 1, 2, \dots)$ is generated by the function

$$F(z) = \frac{1}{f(1/z)} \in \Sigma, \qquad |z| > 1, \tag{4}$$

and the symbol $\{\psi(z)\}_{z=t}^{(n)}$ denotes the n-th partial sum of the Taylor expansion of the function $\psi(z)$ about $z = \infty$ at $z = t$.

Proof. Let $f(z)$ be an arbitrary function of the class S, and let $F(z)$ be the function of the class Σ corresponding to it by (4). Exponentiating equality (1), we introduce the system of functions $\{D_n(z)\}$ $(n = 1, 2, \dots)$:

$$\frac{z - t}{F(z) - F(t)} = \exp\left[\sum_{n=1}^{\infty} A_n(t) z^{-n}\right] = 1 + \sum_{n=1}^{\infty} D_n(t) z^{-n}, \qquad |z| > 1, \quad |t| > 1. \tag{5}$$

Now from (5), by multiplying by $z/(z - t)$, we obtain the expansion

$$\frac{z}{F(z) - F(t)} = 1 + \sum_{n=1}^{\infty} \varepsilon_n(t) z^{-n}, \qquad |z| > |t| > 1, \tag{6}$$

where it is denoted

$$\varepsilon_n(t) = t^n \sum_{k=0}^{n} D_k(t) t^{-k}, \qquad n = 1, 2, \dots; \qquad D_0 = 1, \quad |t| > 1. \tag{7}$$

Alongside (6), for any finite w consider the function $z/[F(z) - w]$ and its Taylor expansion about $z = \infty$. We have:

$$\frac{z}{F(z) - w} = 1 + \sum_{n=1}^{\infty} P_n(w) z^{-n}. \tag{8}$$

It is clear that the coefficients $P_n(w)$ $(n = 1, 2, \dots)$ are polynomials of degree n in w.

If in equality (8) we put $w = F(t)$, where t is an arbitrary finite value from the exterior of the unit circle, then, by uniqueness of the expansion of a regular function in a Taylor series, from (6) and (8) we obtain

$$\varepsilon_n(t) = P_n(F(t)), \qquad n = 1, 2, \dots, \quad |t| > 1. \tag{9}$$

If, however, in equality (8) we put $w = 0$, then for the same reason, from (8) and (4) we shall have:

$$c_{n+1} = P_n(0) \qquad (n = 1, 2, \ldots). \tag{10}$$

Since the function $F(z)$ does not vanish in the domain $|z| > 1$, the interior of any level line C_ρ (C_ρ is the image of the circle $|z| = \rho > 1$ under the mapping by the function $F(z)$), $1 < \rho < \infty$, contains the point $w = 0$, and then, by the maximum-modulus principle for analytic functions, from (9) and (10), replacing n by $n-1$, we derive the inequality

$$|c_n| \le \max_{w \in C_\rho} |P_{n-1}(w)| = \max_{|t|=\rho} |\varepsilon_{n-1}(t)|, \qquad n = 2, 3, \ldots, \quad 1 \le \rho < \infty \tag{11}$$

(the equality sign is possible only for $\rho = 1$; in this case the right-hand side of (11) is understood as the limit as $\rho \to 1$).

Inequality (11), taking (7) into account, can be rewritten as

$$|c_n| \le \rho^{n-1} \max_{|t|=\rho} \left| \sum_{k=0}^{n-1} D_k(t) t^{-k} \right|, \qquad n = 2, 3, \ldots, \quad 1 \le \rho < \infty, \quad D_0 = 1, \tag{12}$$

and in this form it coincides with inequality (3). The theorem is proved.

Inequalities (3) and (2) already make it possible to estimate $|c_n|$ for all n ($n = 2, 3, \ldots$), additionally using only Bunyakovsky' s inequality. To obtain a better estimate, we state in the form of a theorem (without proof) one more inequality, obtained by the author jointly with N. A. Lebedev.

Theorem 2. For any sequence of complex numbers $\{A_k\}$ ($k = 1, 2, \ldots$), for which

$$\sum_{k=1}^{\infty} k |A_k|^2 < \infty,$$

the inequality

$$\sum_{k=0}^{\infty} |D_k|^2 \le \exp\left[\sum_{k=1}^{\infty} k |A_k|^2\right], \tag{13}$$

holds.

where it is denoted:

$$\exp\left[\sum_{k=1}^{\infty} A_k z^{-k}\right] = \sum_{k=0}^{\infty} D_k z^{-k}, \qquad |z| > 1.$$

The equality sign occurs if and only if $A_k = (1/k)c^k$ $(k = 1, 2, ...)$, $|c| < 1$.

We now formulate the final result.

Theorem 3. For a function $f(z) = z + \sum_{n=2}^{\infty} c_n z^n \in S$, the estimate

$$|c_n| < 1.243n \qquad (n = 2, 3, ...). \tag{14}$$

holds.

Proof. Estimating the modulus of the sum in (12) with the aid of Bunyakovsky' s inequality, we shall have:

$$|c_n| \leq \frac{1}{r^{n-1}} \max_{|t|=\rho} \left[\sum_{k=0}^{n-1} |D_k(t)|^2 \sum_{k=0}^{n-1} r^{2k} \right]^{1/2}, \qquad n = 2, 3, \dots, \quad |t| = \rho = \frac{1}{r} > 1. \tag{15}$$

But from inequalities (2) and (13), taking (5) into account, we obtain:

$$\sum_{k=0}^{\infty} |D_k(t)|^2 \leq \frac{1}{1 - r^2}, \qquad |t| = \rho = \frac{1}{r} > 1. \tag{16}$$

Further, the joint consideration of inequalities (15) and (16) leads to the relation

$$|c_n| < \frac{\sqrt{1 - r^{2n}}}{r^{n-1}(1 - r^2)}, \qquad n = 2, 3, \dots, \quad 0 < r < 1. \tag{17}$$

Putting $r^{2n} = e^{-x}$, $0 < x < \infty$, we express the first part in (17) in terms of x. We have:

$$|c_n| < \frac{\sqrt{e^x - 1}}{x} \frac{x/n}{e^{x/2n} - e^{-x/2n}} n, \qquad n = 2, 3, \dots, \quad 0 < x < \infty. \tag{18}$$

Using for $x \in (0, \infty)$ the obvious inequality

$$\frac{x/n}{e^{x/2n} - e^{-x/2n}} < 1 \qquad (n = 2, 3, ...),$$

from (18) we obtain for $|c_n|$ the estimate:

$$|c_n| < \frac{\sqrt{e^x - 1}}{x} n, \qquad n = 2, 3, \dots, \quad 0 < x < \infty.$$

Choosing $x = 1.6$, we arrive at the conclusion of the theorem. The theorem is proved.

If, in Littlewood' s well-known method [2], in the estimates for $|c_n|$ already at the first step–the transition from the modulus of the integral to the integral of the modulus–an excessive factor is imposed (for the Koebe function equal to $e/2$), then inequality (3) does not have this drawback.

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References

[1] I. M. Milin, DAN, 154, No. 2, 264 (1964). [2] J. E. Littlewood, London, Math. Soc., Ser. 2, 23, 481 (1925).

Note: Figure translations are in progress. See original paper for figures.