Quasianalytic Classes of Functions in the Disk

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Abstract

Full Text

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Quasianalytic Classes of Functions in the Disk

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Mathematics

1°. In the present paper, for functions analytic in the unit disk $|z| < 1$ and infinitely differentiable on the circle $|z| = 1$, a problem analogous to the classical Hadamard–Denjoy–Carleman problem of quasianalyticity is solved.

2°. Let $\mathfrak{D}$ be the class of functions $f(z)$ infinitely differentiable in the closed unit disk K ($|z| \leqslant 1$). This means that at each point $z_0 \in K$ and for each function $f(z) \in \mathfrak{D}$ there is an asymptotic expansion

$$f(z) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \quad (x \to z_0,\ z \in K),$$

and, for $|z_0| < 1$, this series obviously has a positive radius of convergence. The boundary values $\tilde{f}(\theta) = f(e^{i\theta})$ ($f \in \mathfrak{D}$) form a class of periodic functions of the real variable θ, which we shall denote by $\widetilde{\mathfrak{D}}$.

Let $\{A_n\}_0^\infty$ be a prescribed nondecreasing sequence of positive numbers. Introduce the following subclasses of $\mathfrak{D}$ and $\widetilde{\mathfrak{D}}$:

1) $\mathfrak{D}\{A_n\}$ is the class of functions $f(z) \in \mathfrak{D}$ for which

$$\max_{z \in K} |f^n(z)| \leqslant CA_n \quad (n = 0, 1, 2, ...), \tag{1}$$

where the constant C depends on $f(z)$.

2) $\widetilde{\mathfrak{D}}\{A_n\}$ is the class of functions $\tilde{f}(\theta) \in \widetilde{\mathfrak{D}}$ for which

$$\max_{-\infty<\theta<\infty} |\tilde{f}^n(\theta)| \leqslant CA_n \quad (n = 0, 1, 2, ...). \tag{2}$$

Definition 1. The class $\mathfrak{D}\{A_n\}$ is called **quasianalytic** if from $f(z) \in \mathfrak{D}\{A_n\}$ and

$$f^{(n)}(z_0) = 0 \quad (n = 0, 1, 2, ...), \tag{3}$$

where z_0 is some point of K, it follows that $f(z) \equiv 0$.

Obviously, (3) can hold for a function $f(z) \in \mathfrak{D}$ not identically equal to zero only when $|z_0| = 1$.

Definition 2. The class $\widetilde{\mathfrak{D}}\{A_n\}$ is called **quasianalytic** if from $\tilde{f}(\theta) \in \widetilde{\mathfrak{D}}\{A_n\}$ and

$$\tilde{f}^{(n)}(\theta_0) = 0 \quad (n = 0, 1, 2, ...), \tag{4}$$

where θ_0 is some point $(-\infty < \theta_0 < \infty)$, it follows that $\tilde{f}(\theta) \equiv 0$.

Remark. The class $\widetilde{\mathfrak{D}}\{A_n\}$, generally speaking, does **not** coincide with the class of functions $f(e^{i\theta})$ that are boundary values of functions from $\mathfrak{D}\{A_n\}$.

Definition 3. We shall say that the sequence $\{A_n\}_0^\infty$ satisfies the Carleman-Ostrowski-Mandelbrojt conditions $(K - O - M)$ if any one of the following equivalent (see (1), p. 29) conditions is fulfilled:

a) If we put $\beta_n = \inf_{k \geq n} A_k^{1/k}$, then

$$\sum^\infty \frac{1}{\beta_n} = \infty. \tag{5}$$

b) If we put $T(r) = \sup_{n \geq 1} r^n / A_n$, then

$$\int^\infty \frac{\log T(r)}{r^2}\, dr = \infty. \tag{6}$$

c) either $\lim_{n\to\infty} A_n^{1/n} < \infty$, or $\lim_{n\to\infty} A_n^{1/n} = \infty$ and

$$\sum^\infty \frac{A_n^c}{A_{n+1}^c} = \infty, \tag{7}$$

where $\{A_n^c\}$ is the convex regularization by means of logarithms (see (1), p. 24) of the sequence $\{A_n\}$.

3°. The following theorem gives the solution of the problem posed.

Theorem 1. *In order that the class $\mathfrak{D}\{A_n\}$ be quasianalytic, it is necessary and sufficient that the sequence $\{\sqrt{A_n}\}$ satisfy the $K - O - M$ conditions.*

The same conditions are necessary and sufficient for the quasianalyticity of the class $\widetilde{\mathfrak{D}}\{A_n\}$.

Remark 1. In the theory of quasianalytic functions one usually considers the broader classes

$$\mathbf{D}\{A_n\} = \bigcup_{k>0} \mathfrak{D}\{k^n A_n\}, \qquad \widetilde{\mathbf{D}}\{A_n\} = \bigcup_{k>0} \widetilde{\mathfrak{D}}\{k^n A_n\}. \tag{8}$$

It can be shown that Theorem 1 is also valid for the classes $\mathbf{D}, \widetilde{\mathbf{D}}$.

Remark 2. The class $\mathfrak{D}\{A_n\}$ is a proper part of the class $\widetilde{D}\{A_n\}$ of all periodic (with period 2π) functions satisfying inequalities (2). From Carleman' s classical result ([2]) it follows that condition (5) is necessary and sufficient for the quasianalyticity of $\widetilde{D}\{A_n\}$.

The condition of Theorem 1 is substantially broader. Thus, for example, the class $\mathfrak{D}\{(n!)^2\}$ is quasianalytic, although $\widetilde{D}\{(n!)^2\}$ does not possess this property.

4°. To prove Theorem 1 it is convenient to map the disk K conformally onto the half-plane $\operatorname{Re} z \geq 0$ in such a way that the point z_0, at which (3) or (4) holds, goes to zero. Under this mapping the sequence $\{A_n\}$ becomes another one, but the fulfillment of the $K-O-M$ conditions for $\{\sqrt{A_n}\}$ is not disturbed. Moreover, instead of inequalities of type (1), (2) it is convenient to consider analogous inequalities for mean-square norms on the straight lines $\operatorname{Re} z = c \geq 0$. Thus we arrive at a theorem which is essentially equivalent to Theorem 1.

Theorem 2. *In order that there not exist a function $f(z) \not\equiv 0$, infinitely differentiable in the half-plane $\operatorname{Re} z \geq 0$ and such that*

$$\max_{0\leq x<\infty} \int_{-\infty}^{\infty} \left|f^n(x+iy)\right|^2 dy \leq A_n^2 \qquad (n=0,1,2,...), \tag{9}$$

$$f^{(n)}(0) = 0 \qquad (n=0,1,2,...), \tag{10}$$

* It is known that the maximum in (9) is attained at $x = 0$.

necessary and sufficient that the sequence $\{\sqrt{A_n}\}$ satisfy the K—O—M conditions.

Proof. Suppose such a function $f(z)$ exists. As is known ([3]), $f(z)$ can be represented in the form

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} g(t)e^{-tz}\, dt \qquad (\operatorname{Re} z \geqslant 0),$$

where from (9) and (10) it follows that

$$\int_0^{\infty} t^{2n}|g(t)|^2\, dt \leqslant A_n^2, \qquad \int_0^{\infty} t^n g(t)\, dt = 0 \quad (n=0,1,2,...). \tag{11}$$

We have arrived at the problem of uniqueness of the solution of a certain Stieltjes-type moment problem, but for measures from L^2. We shall show that problem (11) reduces to the classical Watson problem.*

Consider the function

$$G(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty t^{z-\frac{1}{2}} g(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{(z+\frac{1}{2})\xi} g(e^\xi)\, d\xi \qquad (\operatorname{Re} z > 0).$$

From (11) it follows that $G(z)$ has the following properties:

a) $M(x) = \int_{-\infty}^\infty |G(x+iy)|^2\, dy$ is bounded in every finite interval $0 < x < a$;

b) $M(n) \leqslant A_n^2$ $(n = 0, 1, 2, ...)$;

c) $G\left(\frac{1}{2} + n\right) = 0$ $(n = 0, 1, 2, ...)$.

Hence it is easy to conclude that the function $\Phi(z) = G(z) \sec \pi z$ is analytic in the half-plane $\operatorname{Re} z > 0$, and

$$M_1(x) = \int_{-\infty}^\infty \operatorname{ch}^2 \pi y\, |\Phi(x+iy)|^2\, dy$$

is bounded in every finite interval $(0, a)$, and

$$M_1(n) \leqslant A_n^2 \quad (n = 0, 1, 2, ...). \tag{12}$$

Finally, consider the Fourier transform of $\Phi(z)$

$$\varphi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \Phi(iy) e^{-iy\xi}\, dy. \tag{13}$$

From (12) it follows that $\varphi(\xi)$ is analytic in the strip $|\operatorname{Im} \xi| < \pi$, and its boundary values on the lines $\operatorname{Im} \xi = \pm\pi$ are square-summable. Using the analyticity of $\Phi(z)$, we may shift the path of integration in (13):

$$\varphi(\xi) = \frac{1}{\sqrt{2\pi}} e^{-n\xi} \int_{-\infty}^\infty \Phi(n+iy) e^{-iy\xi}\, dy.$$

By virtue of (12),

$$\int_{-\infty}^\infty |\varphi(\xi+i\eta)|^2 e^{2n\xi}\, d\xi = \int_{-\infty}^\infty |\Phi(n+iy)|^2 e^{2\eta y}\, dy \leqslant$$

$$\leqslant 4 \int_{-\infty}^\infty |\Phi(n+iy)|^2 \operatorname{ch}^2 \pi y\, dy \leqslant 4A_n^2 \quad (-\pi < \eta < \pi;\ n = 0, 1, 2, ...).$$

* The connection between the uniqueness of the Stieltjes moment problem and the Watson problem is known: see, for example, ([4]).

Thus, for every function $f(z) \neq 0$, analytic in the half-plane $\operatorname{Re} z > 0$ and satisfying (9), (10), one can construct $\psi(\zeta) = 1/\varphi(\zeta) \neq 0$, analytic in the strip $|\operatorname{Im}\zeta| < \pi$ and satisfying the conditions

$$\int_{-\infty}^{\infty} |\psi(\xi + i\eta)|^2 e^{2n\xi}\, d\xi \leq A_n^2 \quad (-\pi < \eta < \pi;\ n = 0, 1, 2, \ldots), \tag{14}$$

and conversely. Problem (14) is easily reduced to the classical Watson problem for the strip $|\eta| < \pi$, if one considers the function

$$\psi_\delta(\xi + i\eta) = \int_{\xi}^{\xi+\delta} \psi(\xi + i\eta)\, d\xi$$

($\delta > 0$), which, by virtue of (14), satisfies the inequalities

$$|\psi_\delta(\xi + i\eta)| \leq C A_n e^{-n\xi} \quad (|\eta| < \pi,\ \xi > 0;\ n = 0, 1, 2, \ldots). \tag{15}$$

As is known ([1], p. 55), for the existence of a nonzero solution of problem (15) it is necessary and sufficient that the sequence $\{\sqrt[n]{A_n}\}$ not satisfy the C-O-M conditions. The same is necessary and sufficient for the existence of a nonzero solution of problem (14). The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.