# Aerodynamics

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## Abstract

## Full Text

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# On the Laws of Development and Stabilization of Aerodynamic Disturbances

In recent years, along with the development of the linear theory of the development of aerodynamic disturbances (⁀1), investigations based on nonlinear theory have appeared. In his fundamental work (⁀2), Landau obtained a nonlinear equation for the square of the modulus of the amplitude and pointed out the possibility of the appearance of new regularities in the development of aerodynamic disturbances. Works ([3-]8) are also devoted to the development of nonlinear theory. However, despite a fairly extensive literature and a number of interesting results, the problem as a whole has not been sufficiently developed, since even the basic question of the direction of the process of disturbance development has not been resolved. In the literature there are inaccuracies and, in particular, unfounded assertions concerning the stability of laminar flows with respect to infinitely small disturbances and, at the same time, their instability with respect to disturbances of finite magnitude (⁀9).

In the present work the principal results of the investigations indicated above will be discussed, and it will be shown that Landau' s theory does not make it possible to determine uniquely the direction of the process of disturbance development; in the investigations of Stuart, Watson, and Eckhaus, Landau' s equation was obtained approximately from the equations of hydrodynamics, but in these works it proved impossible to find the unknown constant entering Landau' s equation and determining the laws of development of aerodynamic disturbances; in the author' s investigations a uniformly exact process of successive approximations was constructed, which led to a completely definite law of the development of disturbances in time, containing no unknown constants. In the present work it will also be shown that these results of the author can be extended to more general cases.

Proceeding from the solution of the linearized problem, for sufficiently small $t$ we shall have

$$\psi(x, y, t) = A(t) f_1(x, y). \tag{1}$$

With further increase of $t$, when the nonlinearity of the process begins to manifest itself, but $A$ is still a small quantity, Landau (⁀2) approximately obtained the equation:

$$\frac{d}{dt}|A|^2 = 2\gamma_1|A|^2 - \alpha|A|^4. \tag{2}$$

Here $\gamma_1$ is determined by the linear theory, while $\alpha$ is an unknown constant. As is evident, the character of the development of disturbances will depend essentially on the magnitude and sign of the constant $\alpha$. In the stable region of the flow ($\gamma_1 < 0$): for $\alpha > 0$ all disturbances will decay; for $\alpha < 0$ small disturbances will decay, while disturbances with amplitude greater than $\sqrt{2|\gamma_1/\alpha|}$ will grow without bound.* In the unstable region of the flow ($\gamma_1 > 0$): for $\alpha > 0$ all disturbances will tend toward a flow with amplitude $\sqrt{2|\gamma_1/\alpha|}$; for $\alpha < 0$ all disturbances will grow. Which of these flow regimes is realized in reality this theory does not make it possible to determine. Landau pointed out the possibility of stabilization of aerodynamic disturbances for $\alpha > 0$ in the unstable region and the possibility of the appearance of a new type of instability for $\alpha < 0$ in the stable region (^2). The question of the realization of these possibilities remained open.

* This is often unjustifiably extended to the behavior of finite disturbances.

Stuart' s work ([3]) is also based on expression (1). To determine the square of the modulus of the disturbance amplitude, the energy equation is used:

$$\frac{d}{dt}\iint \frac{{u'}^2 + {v'}^2}{2}\, d\tau = \iint \left(-\overline{u'v'}\right)\frac{\partial U}{\partial y}\, d\tau - \frac{1}{\mathrm{Re}}\iint \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right)^2 d\tau. \tag{3}$$

Expressions (1) and (3) can lead only to a linear equation. Nonlinearity can appear in this case only when the feedback influence of the disturbance on the original flow is taken into account. From the Reynolds equation Stuart finds the approximate expression

$$\partial U/\partial y = -2y + Re\,(\overline{u'v'}). \tag{4}$$

From (1), (3), and (4) the nonlinear equation (2) follows easily. The constant was estimated from linear theory and turned out to be positive. This appears plausible, but cannot be considered justified. Stuart assumes that the form of the linear disturbance is preserved for a sufficiently long time, and it becomes necessary to take into account its influence on the mean motion. In reality, however, higher harmonics will appear much earlier; they may substantially change the entire picture of the flow and, consequently, the magnitude and sign of the coefficient $\alpha$.

In the work ([4]) the stream function is represented as a Fourier series, with amplitudes $\Phi_0, \Phi_1, \Phi_2, \ldots$ for the corresponding harmonics. The mean velocity is equal to $\bar{u} = \partial\Phi_0/\partial y$. For the functions $\bar{u}(y,t), \Phi_1(y,t), \Phi_2(y,t), ...$ a complicated system of equations is obtained. Since in the neighborhood of the neutral curve, according to the data of ([2]), the order of $\Phi_1$ is $\sqrt{\gamma_1}$, the work assumes $\Phi_1 \approx \gamma_1^{1/2}$, $\Phi_2 \approx \gamma_1$, $\Phi_3 \approx \gamma_1^{3/2}$. Then, from the complicated system of equations, as $\gamma_1 \to 0$, there follows a system of three inhomogeneous differential equations for the functions $\bar{u}, \Phi_1, \Phi_2, ....$ The solution of this system is sought in the form

$$\Phi_1 = A(t)\Psi_1(y) + A_{11}\Psi_{11}; \quad \Phi_2 = A^2\Psi_2(y); \quad \bar{u} = 1 - y^2 + |A|^2 f_1(y). \tag{5}$$

Among all solutions of the system of equations for the functions $\Psi_1, \Psi_2, A(t), ...$, which can be obtained by the method of separation of variables, the one is selected for which the amplitude satisfies the equation

$$dA/dt = \gamma_1 A + ik|A|^2 A. \tag{6}$$

This is justified by the fact that equation (6) leads to the already known equation (2), which was obtained in ([3]) from energy considerations. The unknown constant in the equation could not be determined in this work.

In Watson' s work ([5]) the initial assumptions are the same as in ([4]). The stream function is represented as a Fourier series. For the functions $\bar{u}, \Phi_1, \Phi_2, ...$ an infinite system of equations is obtained. The solution is sought (as $|A| \to 0$) in the form

$$\Phi_n(y,t) = A^n(t)\left[\Psi_n(y) + \sum_{m=1}^{\infty} |A|^{2m}\Psi_{mn}(y)\right],$$

$$\bar{u} = 1 - y^2 + \sum_{m=1}^{\infty} |A|^{2m} f_m, \qquad \frac{dA}{dt} = A\sum_{m=0}^{\infty} a_m |A|^{2m}, \tag{7}$$

where $a_0 = \gamma_1$, and $a_n$ are certain constants.

Substituting (7) into the infinite system of equations for $\bar{u}, \Phi_1, \Phi_2, ...$ and equating coefficients of like powers of $|A|^2$, Watson obtains very cumbersome recurrent systems of equations for the set of functions $\Psi_n(y), \Psi_{mn}(y), f_n(y)$, into which the set of unknown constants $a_n$ enters. The work considers in general terms the possibility of a successive and approximate solution of these systems; however, no specific solutions or estimates are given. In the last of the equa-

···of the expansions (7), discarding all constants $a_n$ except $a_0$ and $a_1$, Watson naturally arrives at equation (2) with one unknown constant.

In Eckhaus' work ([6]) the stream function is represented in the form of a Fourier series, and, to simplify the calculations, it is assumed that $\Phi_0 = \varepsilon^2\Psi_0$, $\Phi_1 = \varepsilon\Psi_1$,

$\Psi_2 = \varepsilon^2\Psi_2, ...,$ where $\varepsilon \ll 1$. For the functions $\Psi_m(y,t)$ a complicated recursive system of inhomogeneous equations of the form

$$\left(L_m - \frac{\partial}{\partial t} S_m\right)\Psi_m(y,t) = F_m(y,t). \tag{8}$$

In the corresponding homogeneous and stationary problem, the operator of Orr –Sommerfeld will stand on the left. In the general case the eigenfunctions do not form a complete orthonormal system. Nevertheless, the possibility of solving inhomogeneous equations by the method of eigenfunctions ([10]) is still retained here. This possibility is used in ([6]), and the formal solution is written as

$$\Psi_m(y,t) = \sum_p A_p^{(m)}(t)\varphi_p^{(m)}(y),$$

$$\frac{dA_p^{(m)}}{dt} + \mu_p^{(m)}A_p^{(m)} = -\int_0^1 F_m(\eta,t)\widetilde{\varphi}_p^{(m)}(\eta)\,d\eta. \tag{9}$$

On the basis of (9), the values of the first three functions are estimated:

$$\Psi_0 = |A|^2 q_0(y) + \dots, \qquad \Psi_1 = A\varphi_0^{(1)}(y) + \dots, \qquad \Psi_2 = A^2 q_2 + \dots, \tag{10}$$

where

$$\frac{dA}{dt} + \mu_0^{(1)}A = -\varepsilon^2\beta_0|A|^2A + O(\varepsilon^4), \qquad \beta_0 = \int_0^1 f\widetilde{\varphi}_0^{(1)}\,d\eta. \tag{11}$$

From (11), for the square of the modulus of the amplitude one again obtains the well-known equation (2), which still contains one unknown constant. In Eckhaus' work it is not determined even up to its sign. Nevertheless, the conditions for the appearance of "limiting" solutions of equation (2) are again considered here; this had already been considered by Landau ([11]).

In the author' s work ([7]), the ordinary small-parameter method was applied to the study of solutions of the nonlinear equations of aerodynamics* and it was shown that the simplest particular solution can be written in the form

$$\Psi(x,y,t) = \sum_{n=1}^{\infty} \lambda^n F_n(x,y,t)e^{\gamma_1 nt}, \tag{12}$$

where $F_n(x,y,t)$ are periodic functions of the variables $x$ and $t$, whose determination can be reduced to quadratures. It is established that the stability boundaries and the character of disturbance development in time in successive approximations are determined by the first linear approximation. In the stable region of the flow, $\gamma_1 < 0$, all subsequent approximations will tend to zero. Expansion (12) retains its meaning and possesses uniform accuracy over the entire time interval. The original laminar flow will be stable with respect to infinitesimally small disturbances. Small disturbances of finite magnitude will decay. In the unstable region of the flow, $\gamma_1 > 0$, the first and subsequent approximations will grow without bound in time. Expansion (12) and the first approximation lose their meaning. In this case the classical disturbance method gives no solution.

In the author' s work ([8]), a modified small-parameter method was applied, and it was shown that a particular solution can be written in the form

$$\Psi(x, y, t) = \sum_{n=1}^{\infty} A^n(t) \Phi_n(x, y, \xi), \tag{13}$$

* The author recently learned of Lin' s work ([12]) on the application of this same method to the study of the second approximation for three-dimensional disturbances.

where

$$A(t) = \frac{\lambda}{(1+\lambda)e^{-\gamma_1 t} + \lambda}, \qquad \xi = t - \frac{1}{\gamma_1} \ln(1 - \lambda + \lambda e^{\gamma_1 t}). \tag{14}$$

In the stable region of the flow, $\gamma_1 < 0$, the successive approximations will, as before, decrease. Large disturbances ($\lambda > 1$) will grow with time, but they are not considered in the present work. In the unstable region of the flow, $\gamma_1 > 0$, the successive approximations will be bounded in time. Now the expansion (13) retains its meaning and possesses uniform accuracy over the entire time interval.

The successive approximations will stabilize in time, and their amplitude, independently of the values of $\lambda$, will tend to unity. Thus, here an explicit law of the development and stabilization of the amplitude in time has been obtained, containing no unknown constants (14). If the necessary convergence conditions are satisfied, the expansion (13) will tend to the exact stationary solution of the equations of hydrodynamics.

In works ([8],[9] ) the questions of damping and stabilization of disturbances were considered for particular solutions of the equations of hydrodynamics. We shall show that this can be done in a more general form. We shall seek the solution in the form:

$$\Psi(x, y, t) = \sum_{n=1}^{\infty} \lambda^n \Phi_n(x, y, \xi), \qquad t = \xi + \sum_{n=1}^{\infty} \lambda^n T_n(\xi). \tag{15}$$

Substituting (15) into the equation for the stream function and equating the coefficients of like powers of the parameter $\lambda$, we obtain the recurrent system of equations:

$$\frac{\partial \nabla^2 \Phi_1}{\partial \xi} + L_0(\Phi_1) = 0, \qquad \frac{\partial \nabla^2 \Phi_2}{\partial \xi} + L_0(\Phi_2) = N(\Phi_1 \Phi_1) + \frac{\partial \nabla^2 \Phi_1}{\partial \xi}\frac{dT_1}{d\xi},$$

$$\frac{\partial \nabla^2 \Phi_3}{\partial \xi} + L_0(\Phi_3) = \widetilde{N}(\Phi_1 \Phi_2) + \frac{\partial \nabla^2 \Phi_2}{\partial \xi}\frac{dT_1}{d\xi} + \frac{\partial \nabla^2 \Phi_1}{\partial \xi}\left[\frac{dT_2}{d\xi} - \left(\frac{dT_1}{d\xi}\right)^2\right]. \quad (16)$$

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Putting the auxiliary parameter $\lambda$ equal to unity, we write the conditions for stabilization of the disturbances in the following way:

$$dT_n/d\xi = (dT_1/d\xi)^n = a^n, \qquad -1 < a < 1, \qquad a|_{t\to\infty} \to 1. \quad (17)$$

From expressions (15) and (17) we shall have

$$d\xi/dt = 1 - a, \quad (18)$$

where

$$(d\xi/dt)_{\min} = 0 \text{ as } t \to \infty; \qquad (d\xi/dt)_{\max} \leq 2 \text{ when } t = t_0.$$

These conditions are satisfied, in particular, by a monotonically increasing function $\xi(t)$, varying from the value $\xi = 0$ at $t = 0$ to the value $\xi = \xi_0$ as $t \to \infty$. In this case the expansion (15) has uniform accuracy over the entire time interval and, if the necessary convergence conditions are satisfied, tends to the exact stationary solutions of the equations of hydrodynamics.

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*Note: Figure translations are in progress. See original paper for figures.*