



Soviet-era science, translated into English

Academician L. V. KANTOROVICH, I. V. ROMANOVSKII

1965

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196501.05073>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

CYBERNETICS AND CONTROL THEORY

Academician L. V. KANTOROVICH, I. V. ROMANOVSKII

DEPRECIATION PAYMENTS

UNDER OPTIMAL USE OF EQUIPMENT

The present note is devoted to a mathematical-economic analysis of certain problems relating to the optimal equipping and use of equipment, as well as to the construction, for the corresponding optimal regime, of a system of depreciation deductions and other economic indicators. The problem is considered on comparatively simple models under various simplifying hypotheses, which makes it possible to carry out a fairly complete investigation. The conclusions we obtain, in addition to quantitative characteristics, also make it possible to draw qualitative conclusions about the nature of the optimal solution and the structure of the economic indicators.

For many branches of production, an unevenness of equipment load is characteristic; it changes substantially—cyclically or otherwise—in different periods. Here we consider problems of equipping and using equipment precisely under such conditions. The problem posed is to ensure the planned load with minimal reduced costs for acquiring equipment and to construct indicators that stimulate the implementation of optimal regimes.

1. Let us consider the following model. Suppose that the prescribed load in any time interval $[t, t + \tau]$ is as follows: during $\vartheta_1\tau$ of the time it is single (requiring one machine, which performs work $\vartheta_1\tau$) and during $\vartheta_2\tau$ it is double (requiring two machines, each of which performs work $\vartheta_2\tau$), $\vartheta_1, \vartheta_2 > 0$, $\vartheta_1 + \vartheta_2 \leq 1^*$. To perform this load it is necessary at every moment of time to have at least two machines. A machine may be purchased at any moment of time at price C . A new machine has initial working resource \bar{R} ; the resource of a machine is decreased by ρ when work ρ is performed. A machine with zero resource can no longer be used.

Suppose that at the initial moment ($t = 0$) there are two machines with residual resources R_1 and R_2 ($\bar{R} \geq R_1 \geq R_2 \geq 0$). It is required to determine the moments t_3, t_4, \dots ($t_1 = t_2 = 0$, $t_{k+1} \geq t_k$, $t_n \rightarrow \infty$) for the acquisition of new machines so that there should exist a regime of machine use ensuring fulfillment of the described load^{1**}, and so that the reduced cost

* We specify not a function but a load density, which may be regarded as the

limit (in the weak topology) of the graph of a cyclic load as the cycle length tends to zero. We note the connection of such a formulation of the problem with the time-mixing by point-mixing adopted in dynamic programming (2).

** For this it is necessary and sufficient that there exist measurable functions $g_1(t)$, $g_2(t)$, and $g_3(t)$ of the nonnegative argument t , such that $g_2(t) \neq g_3(t)$,

$$\inf_{\{g_i(t)=k\}} t \geq t_k, \quad i = 1, 2, 3, \quad k = 1, 2, \dots,$$

$$\vartheta_1 \text{mes}\{g_1(t) = k\} + \vartheta_2 \text{mes}(\{g_2(t) = k\} \cup \{g_3(t) = k\}) \leq R_k,$$

where $R_k = \bar{R}$ for $k \geq 3$.

Fulfillment of this conditional load ensures the possibility of fulfilling any real load schedule that it majorizes integrally.

the expenditures for the acquisition of machines over the entire period $[0, \infty)$

$$C \sum_{k=1}^{\infty} e^{-\nu t_k} \quad (1)$$

is then minimal. Here ν is the rate of efficiency (see (1), p. 292). Denote this minimum (whose existence is obvious) by $f(R_1, R_2)$. Put $\vartheta_1 + \vartheta_2 = \Theta$, $\vartheta_2/\Theta = \beta$, $\bar{R}\vartheta_2/(\vartheta_2 + \Theta) = \bar{R}$, $\exp(-\nu\bar{R}/\vartheta_2) = \alpha$.

Using the optimality principle (2), one can obtain for $f(R_1, R_2)$ the following functional equation:

$$f(R_1, R_2) = \min \begin{cases} \min_t e^{-\nu t} f(R_1 - \Theta t, R_2 - \vartheta_2 t), & 0 \leq t \leq \min\left(\frac{R_1}{\Theta}, \frac{R_2}{\vartheta_2}\right), \\ \min_t e^{-\nu t} f(R_1 - \vartheta_2 t, R_2 - \Theta t), & 0 \leq t \leq \min\left(\frac{R_1}{\vartheta_2}, \frac{R_2}{\Theta}\right), \\ C + f(\bar{R}, R_1 + R_2), \end{cases} \quad (2)$$

where the minimum is taken over the following strategies: in the first case q_t —use of the first machine under any load, and of the second only under double load for a time t ; in the second case q'_t —the opposite use; in the third, Q —the purchase of a new machine. The last strategy will be used only when one of the machines is worn out, or when the existing machines are simultaneously fictitiously combined (which means, in fact, their alternate use).

We shall seek optimal behavior in the class of stationary behaviors, and therefore it is sufficient for us to determine the optimal strategy $q(R_1, R_2)$ in any state of the process (R_1, R_2) .

Fig. 1 and Fig. 2

Figure 1: Fig. 1 and Fig. 2

Theorem 1*. If $\alpha \geq \beta$, the behavior \tilde{q} , defined by the following relations, is optimal:

$$q(R_1, R_2) = \begin{cases} Q & \text{if } R_1 + R_2 \leq \tilde{R} \text{ or } R_2 = 0, \\ q_{R_2/\vartheta_2} & \text{if } \Theta R_2 \leq \vartheta_2(R_1 - \tilde{R}) \text{ and } R_2 > 0, \\ q'_t & \text{in all other cases,} \end{cases} \quad (3)$$

where

$$\tilde{t} = \min \left\{ \frac{1}{\vartheta_1(\vartheta_1 + 2\vartheta_2)} [\vartheta_2(\tilde{R} - R_1) + \Theta R_2], \frac{1}{\vartheta_1 + 2\vartheta_2} [R_1 + R_2 - \tilde{R}] \right\}.$$

The proof of optimality of the behavior \tilde{q} is based on a simple generalization of the theorems of dynamic programming (2), connected with approximation in the space of behaviors. When the behavior \tilde{q} is used, the process, after a sufficiently long time T (which can be specified), will be only in states of the form

$$(\tilde{R} + x/\beta, x), \quad (4)$$

where $0 \leq x \leq \tilde{R}$. In Fig. 1 are shown the zones in which the strategies q , q' , and Q are used, and the trajectories of the process in these zones. The dashed line shows the trajectory of the process, starting at the point S with the purchase of a new machine, up to the entry into the steady state (bold line).

In the case $\alpha < \beta$, the formulation of the theorem becomes somewhat more complicated. Here the strategy Q is used when $R_1 + R_2 \leq R' = \tilde{R} \ln \alpha / \ln \beta$ (and, of course, when $R_2 = 0$), the strategy q when $\Theta R_2 \leq \vartheta_2(R_1 - R')$, and the strategy q' in the remaining cases (until entry into another zone). States of the form (4) will still be stable; however, in addition to them, there appears a stable cycle (bold lines in Fig. 2), to which the process comes from any un-

* An analogous result was obtained by V. A. Bulavskii.

of a steady state in a finite, but not uniformly bounded over all states, time.

2. Let now the number of machines required to perform a unit load be equal to n , and for a double load to $2n$. It can be shown that in the case $\alpha \geq \beta$, for any n and any initial state of the process, the regime established after a finite interval of time will be composed

Fig. 1

Fig. 2

of pairs of type (4), i.e., will have the form

$$(\bar{R} + x_1/\beta, \bar{R} + x_2/\beta, \dots, \bar{R} + x_n/\beta, x_1, \dots, x_n),$$

where $\tilde{R} \geq x_1 \geq \dots \geq x_n \geq 0$.

There are grounds to suppose that states of this type will be the only possible steady optimal states for sufficiently large n and in the case $\alpha < \beta$, although we do not have a rigorous justification of this fact.

Let us now pose the problem, under the conditions of such a regime, of determining the value of a machine with any residual resource R , as well as the differential costs of work c_1 and c_2 , respectively under unit and double (peak) load. Under the described steady regime, a machine with resource from \bar{R} to \tilde{R} is used under both types of load, while a machine with resource less than \tilde{R} is used only under peak load. Denote the value of a machine with resource R by $\eta(R)$. Then, from the condition that the purchase of a machine is justified,

$$\eta(R) = \begin{cases} \int_0^{R/\vartheta_2} e^{-\nu t} c_2 \vartheta_2 dt, & \text{for } R \leq \tilde{R}, \\ \int_0^{(R-\tilde{R})/\Theta} e^{-\nu t} (c_1 \vartheta_1 + c_2 \vartheta_2) dt + \int_{(R-\tilde{R})/\Theta}^{(R-\tilde{R})/\Theta + \tilde{R}/\vartheta_2} e^{-\nu t} c_2 \vartheta_2 dt, & \text{for } R \geq \tilde{R}. \end{cases} \quad (5)$$

Since $\eta(\bar{R}) = C$, we have

$$\frac{1-\alpha}{\nu} (c_1 \vartheta_1 + c_2 \vartheta_2) + \frac{\alpha - \alpha^2}{\nu} c_2 \vartheta_2 = C. \quad (6)$$

Finally, for a machine with the boundary resource \tilde{R} , the values given by both formulas in (5) must coincide together with their derivatives, whence

$$\alpha c_2 = c_1 \quad (7)$$

and, finally,

$$c_1 = \frac{C\alpha\nu}{(1-\alpha)(\vartheta_2 + \alpha\Theta)}, \quad c_2 = \frac{C\nu}{(1-\alpha)(\vartheta_2 + \alpha\Theta)}; \quad (8)$$

$$\eta(R) = \begin{cases} \frac{C\vartheta_2}{(1-\alpha)(\vartheta_2 + \alpha\Theta)} (1 - e^{-\nu R/\vartheta_2}), & \text{for } R \leq \tilde{R}, \\ \frac{C\alpha\Theta}{(1-\alpha)(\vartheta_2 + \alpha\Theta)} (1 - e^{-\nu(R-\tilde{R})/\Theta}) + \frac{C\vartheta_2}{\vartheta_2 + \alpha\Theta}, & \text{for } R \geq \tilde{R}. \end{cases} \quad (9)$$

The provision and use of a machine over some interval of time $\Delta\tau$ is associated with two kinds of costs—the securing of the machine in connection with the reduction of its resource during use, and losses associated with the immobilization of funds, accounted for by discounting costs occurring at different times. Accordingly, depreciation charges should be divided into two components: $A_L(R)\Delta\tau$ —for the time during which the machine is made available, and $A_W(R)\Delta R$ —for the use of its resource. Under the optimal regime of use of the machine, these payments must at every moment be compensated by payment for the work performed, and the discounted value of these payments must coincide with the value of the machine. Since, obviously,

$$A_L(R) = \nu\eta(R),$$

$$c_2\vartheta_2 = A_L(R) + \vartheta_2 A_W(R) \quad \text{for } R \leq \tilde{R},$$

$$c_1\vartheta_1 + c_2\vartheta_2 = A_L(R) + \Theta A_W(R) \quad \text{for } R \geq \tilde{R},$$

it follows finally that

$$A_W(R) = \begin{cases} \frac{C\nu}{(1-\alpha)(\vartheta_2 + \alpha\Theta)} e^{-\nu R/\vartheta_2}, & \text{for } R \leq \tilde{R}, \\ \frac{C\nu\alpha}{(1-\alpha)(\vartheta_2 + \alpha\Theta)} e^{-\nu(R-\tilde{R})/\Theta}, & \text{for } R \geq \tilde{R}. \end{cases} \quad (10)$$

Let us give one more way of writing $A_W(R)$. Let $t(R)$ be the remaining period of use of a machine with resource R under the optimal regime,

$$t(R) = \begin{cases} R/\vartheta_2, & \text{for } R \leq \tilde{R}, \\ (R - \tilde{R})/\Theta + \tilde{R}/\vartheta_2, & \text{for } R \geq \tilde{R}. \end{cases}$$

Then

$$A_W(R) = c_2 e^{-\nu t(R)}. \quad (11)$$

This formula can be given an economic interpretation that makes it possible to construct its analogues for any types of stationary load and for the case of a decline in the operating qualities of the machine as its resource decreases. The case in which the resource also decreases during periods when the machine is not in use does not introduce any substantial changes into the formulas.

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

Leningrad State University
named after A. A. Zhdanov

Received
13 II 1965

REFERENCES

1. L. V. Kantorovich, *Economic Calculation of the Best Use of Resources*, Publishing House of the Academy of Sciences of the USSR, 1959.
2. R. Bellman, *Dynamic Programming*, Foreign Literature Publishing House, 1960.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.