ON (ω, p)-MAPPINGS OF TOPOLOGICAL SPACES

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Abstract

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MATHEMATICS

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ON (ω, p)-MAPPINGS OF TOPOLOGICAL SPACES

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1. Definition and the main lemma*

Basic definition. Let ω be some fixed covering of a space X. A continuous mapping $f : X \to Y$ of the space X onto the space Y will be called an (ω, p)-**mapping** if, for every point $y \in Y$, there exist a system $\omega_y = \omega_{f^{-1}y} \subseteq \omega$ with property p in its body $\tilde{\omega}_y$, and a neighborhood V_y of the point y in Y, such that

$$f^{-1}V_y \subseteq \tilde{\omega}_y$$

(from which, in particular, it follows that ω_y covers the set $f^{-1}y$).**

Main lemma. Let $f : X \to Y$ be, for a given covering ω of the space X, a continuous (ω, p)-mapping of the space X onto a paracompact (respectively, strongly paracompact) space Y, where p is the property of local finiteness (respectively, countability). Then into the covering ω of the space X one can inscribe a locally finite (respectively, star-countable) covering ω'.

Proof. For each point $y \in Y$, choose a subsystem $\omega_y \subseteq \omega$ with property p (i.e., a subsystem locally finite in the body $\tilde{\omega}_y$, respectively countable) covering the set $f^{-1}y$, and a neighborhood V_y of the point $y \in Y$ in such a way that

$$f^{-1}V_y \subseteq \omega_y.$$

Then $\Omega = \{V_y\}$ is an open covering of the space Y (here y runs through all points of the space Y). Into the covering Ω we inscribe a locally finite covering $\Omega' = \{V'_\alpha\}$, if the space Y is paracompact, and a star-finite one if Y is strongly paracompact. Now, for each element $V'_\alpha \in \Omega'$, take some set $V_{y_\alpha} \in \Omega$ containing the set V'_α. To this V_{y_α} there corresponds a subsystem $\omega_{y_\alpha} \subseteq \omega$ with property p, such that $f^{-1}V_{y_\alpha} \subseteq \tilde{\omega}_{y_\alpha}$. Now denote by ω'_α the system of open sets in X obtained by intersecting the elements of the system ω_{y_α} with the open set $f^{-1}V'_\alpha$ in X. Consider the system

$$\omega' = \bigcup_{\alpha} \omega'_{\alpha}.$$

This system ω' is inscribed in the covering ω and is a locally finite covering of the space X, if p is the property of local finiteness (i.e., all ω_y are locally finite in $\tilde{\omega}_y$), Y is paracompact, and is a star-countable covering of the space X, if p is the property of countability and Y is strongly paracompact. The lemma is proved.***

* In this paper all spaces are assumed to be regular, and all mappings continuous.

** The body $\tilde{\omega}_A$ of the system ω_A is, as usual, the union of the elements contained in ω_A. The system ω_A is locally finite in the body $\tilde{\omega}_A$ (in the whole space X) if, for every point $x \in \tilde{\omega}_A$ (for every point $x \in X$), there exists a neighborhood $Ux \subseteq X$ that intersects only finitely many elements of the system ω_A. The letter p may mean the property of local finiteness of ω_A in $\tilde{\omega}_A$, star-finiteness, or simply countability or finiteness.

*** We also note that ω' will be a star-finite covering of the space X, if p is the property of star-finiteness and the space Y is strongly paracompact.

2. Some consequences of the main lemma.

Proposition 1. *In order that a space X be paracompact, it is sufficient that for every open covering ω of this space there exist a continuous (ω, p)-mapping f of the space X onto a paracompact Y_ω. The property p denotes local finiteness.*

Remark. It is not hard to verify that a one-to-one perfect mapping f of a space X onto a space Y will always be an (ω, p)-mapping for every open covering ω of the space X, and here p will even be the property of finiteness. From this remark we obtain the known theorem (see, for example, ([5])): if f is a one-to-one perfect mapping of a space X onto a paracompact Y, then the space X is also paracompact.

Proposition 2. *Suppose that in the space X there is a countable refining system $\mathfrak{A} = \{\omega_i\}$ of open coverings. Suppose further that for each i there exists a continuous (ω_i, p)-mapping of this space onto some paracompact Y_i, where p is the property of local finiteness. Then the space X is necessarily metrizable.*

Proposition 3. *In order that a space X be strongly paracompact, it is sufficient that for every open covering ω of this space there exist a continuous (ω, p)-mapping f of the space X onto a strongly paracompact space Y_ω. Here p may denote either the property of countability or the property of star-finiteness.*

Let us note that a closed S-mapping** f of a space X onto a space Y will always be an (ω, p)-mapping for any open covering ω of the space X, where p is the property of countability. Therefore from our Proposition 3 it follows that

Proposition 4. *If f is a closed S-mapping of a space X onto a strongly paracompact space Y, then the space X is also strongly paracompact.*

3. Fully paracompact spaces.

A space X is called **fully paracompact** if into every one of its open coverings ω one can weakly inscribe*** a σ-star-finite**** covering ω'. Fully paracompact spaces are necessarily paracompact; at the same time there exist fully paracompact spaces that are not strongly paracompact (see ([4])). Fully paracompact metrizable spaces are also called **strongly metrizable**—these are spaces in which there exists a σ-star-finite base.

Proposition 5. *In order that a normal space X be fully paracompact, it is sufficient (and, obviously, necessary) that into every one of its open coverings ω one can weakly inscribe a σ-star-countable covering ω'.*

From this proposition and from the main lemma it follows:

Proposition 6. *Each of the following properties of a space X is sufficient (and necessary) for its strong metrizability:*

A. *The space X has a σ-star-countable base.*

B. *The space X is metrizable and in it there is such a base $\mathfrak{B} = \bigcup_{i=1}^{\infty} \omega_i$,*

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* A countable system $\mathfrak{A} = \{\omega_i\}$ of open coverings of a space X is **refining** if for any point x of this space and any neighborhood Ux of it there is a covering $\omega_i \in \mathfrak{A}$ such that the sum $\Gamma_i x$ of its elements containing the point x is contained in Ux (in connection with this concept see the papers ([2],[3],[8])).

** A continuous mapping $f : X \to Y$ is called an S**-mapping** if the inverse image of every point $y \in Y$ is a finally compact space (see the papers ([6–8])).

*** A covering ω is **weakly inscribed** in a covering ω of a space X if there exists a subcovering ω'' of the covering ω' inscribed in ω.

**** A system $\mathfrak{B}$ of subsets of a space X is called **σ-star-finite** (σ-star-countable or σ-locally finite)) if the system is the union of a countable number of star-finite (star-countable or locally finite) coverings of this space X.

where each ω_i is an open cover, such that for each i there exists an (ω_i, p)-mapping* $f_i : X \to Y_i$ onto a strongly paracompact Y_i.

B. In the space X there is a countable refining system $\mathfrak{A} = \{\omega_i\}$ of open covers, and for each i there exists an (ω_i, p)-mapping of the space X onto a strongly paracompact Y_i.

4. Mappings into Baire space.** The following arguments are adjacent to the work of Yu. M. Smirnov ([6]).

A system of sets $\eta = \{U_\lambda\}$ of a star-finite cover ω is called **connected** (see ([1,6])) if for every pair U, U' of sets from η there exist sets $U_1, \dots, U_k$ from η such that $U_i \cap U_{i+1} \neq \Lambda$ for $i = 1, 2, \dots, k-1$, and $U_1 \cap U \neq \Lambda$, $U_k \cap U' \neq \Lambda$. A maximal connected subsystem η of the system ω is called a **component** of the system ω. It should be noted that every component of a star-finite system ω is always at most countable, and the body of every component, i.e. the sum of the elements it contains, is an open-and-closed set of the space X (see ([1,6])). Let the components of the star-finite cover ω be η_α, where α are distinct indices, and let $\tilde{\eta}_\alpha$ be the bodies of these components. By D_ω denote the set of these distinct indices, endowed with the discrete topology. Obviously, D_ω is topologically contained in the Baire space B^τ, where τ is the cardinality of the set D_ω. Let, further, $x \in X$ be an arbitrary point. Since the system Ω of all $\tilde{\eta}_\alpha$ is a disjoint open-and-closed cover of the space X, there is a unique set $\tilde{\eta}_\alpha$ containing the point x. We obtain a mapping f_ω of the space X onto the discrete space $D_\omega = \{\alpha\}$: $f_\omega x = \alpha$, where α is the index of the set $\tilde{\eta}_\alpha \ni x$. This mapping is continuous and, moreover, is an (ω, p)-mapping, where p is the property of countability. Thus, we have proved

Lemma 1. If ω is a star-finite cover of the space X, then there exists a continuous (ω, p)-mapping of this space X into Baire space.

Let, further, the space X have a countable set $\mathfrak{A}$ of star-finite covers ω_i. We proceed as follows: in each cover ω_i we consider its components $\eta_{\alpha i}$, the bodies $\tilde{\eta}_{\alpha i}$ of these components (which are open-and-closed sets of the space X), and also the open-and-closed covers Ω_i of the space X, consisting of the open-and-closed sets $\tilde{\eta}_{\alpha i}$ for fixed i. Let now x be an arbitrary point of X. For each i this point belongs to the body $\tilde{\eta}_{\alpha i}$ of some, and moreover unique, component $\eta_{\alpha i}$ of the cover ω_i, i.e. to a unique element of the open-and-closed cover Ω_i. To this point x we put in correspondence the countable sequence $\{\alpha i\}$ of indices, i.e. the point of the space

$$B^\tau = \prod_{i=1}^{\infty} D_{\omega_i},$$

where τ is the greatest cardinality of the sets D_{ω_i}. In this way we have obtained a mapping f of the space X into the space B^τ. This mapping is continuous and, moreover, is an (ω_i, p)-mapping for every i (p is the property of countability). In addition

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* Beginning from this point, p is the property of countability.

** The space B^τ may be defined, for example, as follows: let W be some set of indices of cardinality τ. The points of the space B^τ, by definition, are countable sequences $(\alpha_1, \dots, \alpha_i, \dots)$ of these indices from W. If $x = (\alpha_1, \dots, \alpha_i, \dots)$ and $y = (\beta_1, \dots, \beta_i, \dots)$ are two points of B^τ, then we put $\rho(x, y) = 1/k(x, y)$, where $k(x, y)$ is the first natural number i such that $\alpha_i \neq \beta_i$. The space B^τ, thus, is a metric space of weight τ and dimension $\dim B^\tau = 0$, containing (in the sense of topological embedding) all metrizable spaces of weight τ and dimension dim equal to zero. The space B^τ may be defined also as the topological product of a countable number of discrete spaces D_i of cardinality τ. On Baire space see ([6,8,9]).

therefore, every set η_{a_i} will be marked* for the mapping f. Thus, it has been proved:

Lemma 2. If in the space X there exists a countable set $\mathfrak{A}$ of star-finite covers ω_i, then there exists a continuous mapping into Baire space B^τ which is an (ω_i, p)-mapping for every i.

From Lemmas 1 and 2 the following propositions immediately follow:

Proposition 7. For every open cover ω of a strongly paracompact space X there exists a continuous (ω, p)-mapping of it into Baire space (p is the property of countability).

Proposition 8. Let the space X be strongly metrizable, and let $\mathfrak{B}$ be its base, decomposable into the sum of a countable number of star-finite covers ω_i. Then there exists a continuous (ω_i, p)-mapping (for every i) f of this space X into Baire space B^τ; moreover, f automatically turns out to be an S-mapping.

5. On quasicomponents and quasimonotone mappings. A closed set A of a topological space X is called **quasiconnected** if, for every open-and-closed set $U \subseteq X$ for which $U \cap A \neq \Lambda$, necessarily $A \subseteq U$. A maximal quasiconnected set A is called a **quasicomponent** of the space X. A continuous mapping $f : X \to Y$ is called **quasimonotone** if for every point $y \in Y$ the set $f^{-1}y$ is quasiconnected in X. We note that every quasicomponent of a fully paracompact space X, and every quasiconnected subspace of a fully paracompact space X, is finally compact.

The following basic result holds:

Proposition 9.** Let $f : X \to Y$ be a quasimonotone quotient mapping of a strongly paracompact (respectively, fully paracompact) space X onto a space Y. Then the space Y is also strongly paracompact (respectively, fully paracompact). In particular, the decomposition space Z of a strongly paracompact (respectively, fully paracompact) space X into its quasicomponents will be strongly paracompact (respectively, fully paracompact).

Remark. Every quasimonotone quotient mapping f of a space X onto a space Y will necessarily be an (ω, p)-mapping for every star-finite cover ω of the space X; if the space X is strongly paracompact, then the quotient quasimonotone mapping $f : X \to Y$ will be an (ω, p)-mapping for every open cover ω of the space X. We note also that a quasimonotone quotient mapping f of a strongly paracompact (and also of a fully paracompact) space X onto a space Y will automatically be an S-mapping. Finally, in the case when all open-and-closed sets $U \subseteq X$ are marked for the quotient mapping $f : X \to Y$, the mapping f will, in addition, be quasimonotone.

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* A set $M \subseteq X$ is called **marked** for a mapping $f : X \to Y$ if $M = f^{-1}M$.

** Cf. this proposition with the main theorem in ([10]). A mapping $A : X \to Y$ is called **quotient** if a set $V \subseteq Y$ is open if and only if the set $f^{-1}V$ is open in the space X.

Note: Figure translations are in progress. See original paper for figures.