

**ABSENCE IN THE  
SPACES  $(C)$ ,  $(L_p)$ ,  
AND  $(W_p^1)$   $(1 \leq p < 2)$  OF ANALOGUES  
OF THE ENERGY  
INEQUALITY FOR THE  
STRING EQUATION  
WITH BOUNDED  
LEADING  
COEFFICIENT**

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**Abstract**

**Full Text**

**MATHEMATICS**

**E. B. BYKHOVSKII**

**ABSENCE IN THE SPACES  $C$ ,  $L_p$ , AND  $W_p^1$   
( $1 \leq p < 2$ ) OF ANALOGUES OF THE ENERGY  
INEQUALITY FOR THE STRING EQUATION  
WITH BOUNDED LEADING COEFFICIENT**

*(Presented by Academician V. I. Smirnov, January 23, 1965)*

As is known, for the Cauchy problem

$$v_{tt} = [a(x)v_x]_x; \quad v|_{t=0} = \varphi(x); \quad v_t|_{t=0} = \psi(0) \quad (1)$$

when  $a(x)$  is, say, piecewise smooth and satisfies the inequality

$$0 < \beta \leq a(x) \leq \gamma,$$

there is an estimate of

$$\int_x [v_t^2 + v_x^2] dx$$

by the same integral at  $t = 0$  over an interval of the  $x$ -axis determined by the course of the characteristics of equation (1), with constant  $K(\beta, \gamma)$ . An analogous estimate holds for

$$\int_x v^2 dx.$$

If  $K$  is allowed also to depend on the smoothness indices of  $a(x)$ , then analogous estimates also hold for the norms  $W_p^1$  and  $\mathcal{L}_p$  ( $1 \leq p < 2$ ).

Our purpose is to show that the latter estimates do not hold with  $K(\beta, \gamma)$ . Counterexamples will be constructed for

$$\psi(x) \equiv 0. \quad (2)$$

The functions  $\varphi(x)$  and  $a(x)$ , and consequently also  $v(x, t)$ , will henceforth be assumed  $l$ -periodic in  $x$ .

By  $\|f(x)\|$  we shall mean any one of the norms:

$$\max_{0 \leq x \leq l} |f(x)|, \quad \left\{ \int_0^l |f(x)|^p dx \right\}^{1/p}, \quad \left\{ \int_0^l |f'(x)|^p dx \right\}^{1/p}. \quad (3)$$

**Lemma 1.** *If in problem (1), (2), for one of the indicated norms, for some  $t_0$  the estimate*

$$\|v(x, t_0)\| \leq K(\beta, \gamma)\|v(x, 0)\|$$

*holds, then this estimate holds uniformly in  $t \in [A_{t_0}, \infty)$ ,  $A_{t_0} > 0$ .*

The proof is based on the fact that the function  $v_\alpha(x, t) = v(\alpha x, \alpha t)$  is a solution of (1), (2) with  $a(\alpha x)$  and  $\varphi(\alpha x)$ , while  $0 < \beta \leq a(\alpha x) \leq \gamma$ , and on the fact that for periodic  $f(x)$

$$\|f(\alpha x)\| = \|f(x)\| [\alpha^m + O(1/\alpha)] \quad (\alpha \geq 1),$$

where  $O(1/\alpha)$  is written for large  $\alpha$  and is independent of  $f(x)$ .

Thus, counterexamples will be constructed if one can find a sequence  $\varphi_n(x)$  with bounded  $\|\varphi_n(x)\|$  and such an  $a(x)$  that for  $v_n(x, t)$  one has

$$\sup_{\substack{1 \leq n < \infty \\ 0 \leq t < \infty}} \|v_n(x, t)\| = \infty.$$

In what follows we shall take  $\varphi(x)$  to be odd  $2\pi$ -periodic functions, and  $a(x)$  to be an even  $2\pi$ -periodic function. Then  $v(x, t)$ , for  $0 \leq x \leq \pi$ , will be the solution of the initial-boundary-value problem with boundary conditions  $v(0, t) = v(\pi, t) = 0$ .

For  $v(x, t)$  we have:

$$v(x, t) = \sum_{k=1}^{\infty} c_k \cos \lambda_k t y_k(x),$$

where  $y_k(x)$  are the eigenfunctions of the problem

$$\mathcal{L}y = [a(x)y']' = -\lambda y, \quad y(0) = y(\pi) = 0; \quad (4)$$

$c_k$  is the coefficient in the expansion

$$\varphi(x) = \sum_{k=1}^{\infty} c_k y_k(x).$$

Using the asymptotics of  $y_k(x)$  and adding and subtracting

$$\sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} c_k \cos kt \sin k\xi,$$

we obtain

$$\begin{aligned}
 v[x(\xi), t] &= \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} c_k (\cos \lambda_k t - \cos kt) \sin k\xi \\
 &+ \frac{1}{2} a^{-1/4}[x(\xi)] a^{1/4}[x(\xi + t)] \varphi[x(\xi + t)] \\
 &+ \frac{1}{2} a^{-1/4}[x(\xi)] a^{1/4}[x(\xi - t)] \varphi[x(\xi - t)] + \sum_{k=1}^{\infty} \frac{c_k}{k} \eta_k(\xi),
 \end{aligned} \tag{5}$$

where

$$\xi = \frac{\pi}{h} \int_0^x a^{-1/2}(z) dz, \quad h = \int_0^{\pi} a^{-1/2}(z) dz,$$

and the  $\eta_k$  are uniformly bounded with respect to  $k$  and  $\xi$ .

An analogous expression can be written for  $v_x[x(\xi), t]$ .

**Lemma 2.** *Let an infinite sequence  $\sigma$  of zeros and ones be given arbitrarily. Then there exists a strictly positive  $a(x) \in C_2[0, \pi]$  such that the spectrum  $\lambda_k$  of problem (4) has the following property: for every natural  $n$  there is a  $t_n > 0$  such that the numbers*

$$-\frac{1}{2} (\cos \lambda_{4k} t_n - \cos 4k t_n) \quad (k = 1, 2, \dots, n)$$

are equal, respectively, to the first  $n$  terms of the sequence  $\sigma$ .

In the proof, first the  $\lambda_k$  are chosen in a suitable way according to  $\sigma$ , and then  $a(x)$  is constructed as the solution of the inverse Sturm-Liouville problem <sup>(1)</sup>. It is not hard to see that  $t_n \rightarrow +\infty$  as  $n \rightarrow \infty$ .

Let us consider in detail the case of the space  $C$ . Let

$$S[f] = \sum_{k=1}^{\infty} c_k \sin k\xi$$

be a series converging uniformly but not absolutely. We shall consider the series

$$S[f(4\xi)] = \sum_{k=1}^{\infty} c_k \sin 4k\xi,$$

whose convergence character is the same, and denote its sum by  $S(\xi)$ , and by  $\sigma S_n(\xi)$  the partial sum thinned out by the sequence  $\sigma$ , respectively. One may assert that  $|S_n(\xi)|$  are uniformly bounded, but if  $\xi_0$  is a point of nonabsolute convergence, then there exists a  $\sigma$  for which

$$\sigma S_n[f(4\xi_0)] \rightarrow \infty.$$

Using the indicated  $\sigma$ , we construct  $a(x)$  as stated in Lemma 2, taking the  $\lambda_k$  with indices not divisible by 4 arbitrarily, provided only that the inverse problem

can be solved (for example,  $\lambda_k = k$ ). In problem (1), (2) we take

$$\varphi_n(x) = \sum_{k=1}^n c_k y_{4k}(x).$$

Then  $|\varphi_n(x)|$  are uniformly bounded with respect to  $x$  and  $n$  (by virtue of the asymptotics of  $y_k(x)$ ).

Let  $v_n(x, t)$  be the solution of problem (1), (2) with the indicated  $a(x)$  and  $\varphi_n(x)$ . Using (5), it is not hard to verify that

$$\sup_{n, x, t \in [0, \infty)} |v_n(x, t)| = \infty,$$

which completes the construction of the example. Examples for the remaining norms are constructed analogously, but more complicated properties of trigonometric-

series. For  $\mathcal{L}_p$  one takes the series

$$S[f(4\xi)] = \sum_{k=1}^{\infty} c_k \sin 4k\xi$$

such that: a)  $S_n(\xi) \rightarrow f(4\xi)$  in  $\mathcal{L}_p$ , and b) there exists a  $\sigma$  such that

$$\|\sigma S_n\|_{\mathcal{L}_1} \xrightarrow{n \rightarrow \infty} \infty.$$

The existence of such a series is guaranteed by

**Lemma 3.** Let

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx); \quad f \in \mathcal{L}_p \quad (1 < p)$$

be such that

$$\sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \infty$$

(i.e.  $p < 2$ ).

There exists such a  $\sigma$  that

$$\|\sigma S_n\|_{\mathcal{L}_1} \xrightarrow{n \rightarrow \infty} \infty.$$

The proof is based on theorems from (2) (vol. 1, pp. 214, 148, 266).

For the space  $W_p^1$  one takes the series

$$\sum_{k=1}^{\infty} \frac{1}{k^{1+\beta}} \sin kx,$$

where

$$1 - \frac{1}{p} < \beta < \frac{1}{2}.$$

It converges absolutely and uniformly to some  $f(x)$ , while the differentiated series

$$\sum_{k=1}^{\infty} \frac{1}{k^{\beta}} \cos kx$$

converges uniformly outside any neighborhood of  $x = 0$  ((<sup>2</sup>), vol. 1, p. 4). Its sum  $f'(x)$  in a neighborhood of  $x = 0$  behaves like  $1/x^{1-\beta}$  ((<sup>2</sup>), vol. 1, p. 70). Consequently,  $f'(x) \in \mathcal{L}_p$  ( $p < 2$ ); the series converges to it in  $\mathcal{L}_p$ , and

$$\sum_{k=1}^{\infty} \frac{1}{k^{2\beta}} = \infty,$$

after which Lemma 3 is used.

Thus, the following has been proved.

**Theorem.** Whatever  $t_0 > 0$  is prescribed, in the Cauchy problem (1), (2) one can choose a sequence of coefficients  $a_{(n)}(x)$  and initial data  $\varphi_{(n)}(x)$ , satisfying uniformly in  $n$  the condition

$$0 < \beta \leq a_{(n)}(x) \leq \gamma; \quad \|\varphi_{(n)}(x)\| \leq \text{const} \quad (1 \leq p < 2),$$

so that, for the corresponding solutions  $v_{(n)}(x, t)$  of problem (1), (2), one has not only

$$\|v_{(n)}(x, t_0)\|_p \xrightarrow{n \rightarrow \infty} \infty,$$

but even

$$\|v_{(n)}(x, t_0)\|_1 \xrightarrow{n \rightarrow \infty} \infty.$$

All functions involved here are  $2\pi$ -periodic in  $x$ , and by  $\|f(x)\|_p$  is meant one of the norms (3); moreover, in the case of  $C$  the subscript is superfluous.

Leningrad State University  
named after M. V. Lomonosov

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## References

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- <sup>2</sup> A. Zygmund, *Trigonometric Series*, 2nd ed., 1, 2, Cambridge, 1959.

*Note: Figure translations are in progress. See original paper for figures.*

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