SOME QUESTIONS OF APPROXIMATION ON SETS OF THE COMPLEX PLANE

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1965

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Abstract

Full Text

SOME QUESTIONS OF APPROXIMATION ON SETS OF THE COMPLEX PLANE

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(Presented by Academician I. N. Vekua on 18 I 1965)

Let E be a compact set of the complex plane Z; let $\mu(e)$ be a finite positive measure defined on the Borel subsets e of the compact set E; $L_p(\mu)$, $p \geqslant 1$, is the space of functions $\omega(z)$ satisfying the condition

$$\|\omega\| = \left\{ \int_E |\omega(z)|^p \, d\mu \right\}^{1/p} < \infty.$$

We shall call the measure μ **reduced** with respect to the set E when the measure of every nonempty portion of E is positive.

Recall that a compact set E has zero analytic capacity $\Omega(E) = 0$ if in the complement $Z \setminus E$ there are no bounded analytic functions different from constants.

In the works of S. Ya. Khavinson (^{1}) and V. P. Khavin (^{2}) the following was shown.

Every continuous function $\Phi(z)$ on a set E of zero analytic capacity can be approximated by such a sequence of rational functions

$$\sum^{(k)} \frac{\lambda_j^{(k)}}{z - a_j^{(k)}}$$

with poles outside E, that

$$\max_{z \in E} \left| \Phi(z) - \sum^{(k)} \frac{\lambda_j^{(k)}}{z - a_j^{(k)}} \right| \xrightarrow[k \to \infty]{} 0 \tag{1}$$

and, at the same time,

$$\sum^{(k)} |\lambda_j^{(k)}| \xrightarrow[k \to \infty]{} 0. \tag{2}$$

Conversely, if on a compact set E, about which it is additionally assumed that its length (in the sense of Painlevé, see (^{1})) is finite, the constant one can be approximated by rational fractions in such a way that conditions (1) and (2) are fulfilled, then $\Omega(E) = 0$.

It follows from this that on a set E, $\Omega(E) = 0$, for every measure μ and every $\omega(z) \in L_p(\mu)$, $p \geqslant 1$, there exists a sequence of fractions

$$\sum^{(k)} \frac{\lambda_j^{(k)}}{z - a_j^{(k)}}$$

with poles outside E, satisfying the relations

$$\int_E \left| \omega(z) - \sum^{(k)} \frac{\lambda_j^{(k)}}{z - a_j^{(k)}} \right|^p d\mu \xrightarrow[k\to\infty]{} 0, \tag{3}$$

$$\sum^{(k)} |\lambda_i^{(k)}| \xrightarrow[k\to\infty]{} 0. \tag{4}$$

Let us pose the converse problem. Suppose that, for some measure μ (which, of course, must be regarded as reduced), conditions (3) and (4) hold for all $\omega(z) \in L_p(\mu)$, $p \geqslant 1$. Does it follow from this that $\Omega(E) = 0$, i.e., are (3) and (4) not only necessary but also sufficient conditions for $\Omega(E) = 0$? We give a negative answer to this question.

Theorem 1. For every nowhere dense compact set E there exists a reduced measure μ such that, for all $p \geq 1$, for any function $\omega(z) \in L_p(\mu)$ one can construct a sequence of aggregates $\sum^{(k)} \frac{\lambda_j^{(k)}}{z - a_j^{(k)}}$, for which

$$\int_E \left| \omega(z) - \sum^{(k)} \frac{\lambda_j^{(k)}}{z - a_j^{(k)}} \right|^p d\mu \xrightarrow[k\to\infty]{} 0, \qquad \sum^{(k)} |\lambda_j^{(k)}| \xrightarrow[k\to\infty]{} 0.$$

We shall give the idea of the proof. Let $K = \{z_n\}_1^\infty$ be a countable everywhere dense subset of E; let $\omega_n(z)$ be the characteristic function of the set consisting of the single point z_n. From the fact that $\Omega(\{z_1, z_2, \ldots, z_k\}) = 0$ and the set E is nowhere dense, for each $\omega_n(z)$, $n = 1, 2, \ldots$, there is a fraction

$$\sum^{(k)} \frac{\lambda_j^{(n,k)}}{z - a_j^{(n,k)}}$$

with poles outside the set E, satisfying the conditions:

$$\left|\omega_n(z) - \sum^{(k)} \frac{\lambda_j^{(n,k)}}{z - a_j^{(n,k)}}\right| \le \frac{1}{2^{k-1}} \quad \text{for } z = z_1, \dots, z_k,$$

$$\sum |\lambda_j^{(n,k)}| \le \frac{1}{2^{k-1}}.$$

Denote

$$\max_{z\in E}\left|\omega_n(z) - \sum^{(k)} \frac{\lambda_j^{(n,k)}}{z - a_j^{(n,k)}}\right| = M_k^{(n)}, \qquad n = 1, 2, \dots$$

Since $\omega_n(z_1) = \omega_n(z_2) = \cdots = \omega_n(z_k) = 0$, $n = k+1, k+2, \dots$, the function

$$\sum^{(k)} \frac{\lambda_j^{(n,k)}}{z - a_j^{(n,k)}}$$

may be taken to be the same for all $\omega_n(z)$, beginning with $n = k+1$. Therefore there exists a finite

$$\max_{n=1,2,\dots}\{M_k^{(n)}\} = \max\left\{M_k^{(1)},\ M_k^{(2)},\ \dots,\ M_k^{(k)},\ \max_E\left|1 - \sum^{(k)} \frac{\lambda_j^{(k+1,k)}}{z - a_j^{(k+1,k)}}\right|,\right.$$

$$\left.\max_E\left|\sum^{(k)} \frac{\lambda_j^{(k+1,k)}}{z - a_j^{(k+1,k)}}\right|\right\} = M_k.$$

Put

$$\mu(z_1) = 1, \qquad \mu(z_{k+1}) = \min\left[\frac{\mu(z_k)}{2}, \frac{1}{2^k (M_k)^k}\right], \qquad k = 1, 2, \dots$$

In this way we obtain a nonzero measure on the countable subset $K \subset E$; next we extend it to the whole plane Z in the usual manner:
$\mu(e) = \mu(e \cap K)$.

It can be verified that the measure μ satisfies the conditions of our theorem. At the same time, if E has finite length, then the fulfillment of conditions (3) and (4) for all measures μ is sufficient in order that $\Omega(E) = 0$. More precisely, the following theorem holds:

Theorem 2. Let E have finite length. If, whatever the measure μ may be, there is a sequence

$$\sum^{(k)} \frac{\lambda_j^{(k)}}{z - a_j^{(k)}}$$

(to each measure μ there corresponds its own sequence) satisfying the relations

$$\int_E \left| 1 - \sum^{(k)} \frac{\lambda_j^{(k)}}{z - a_j^{(k)}} \right| d\mu \xrightarrow[k\to\infty]{} 0,$$

$$\sum^{(k)} |\lambda_j^{(k)}| \xrightarrow[k\to\infty]{} 0,$$

then $\Omega(E) = 0$.

The measure μ constructed in Theorem 1 had as its essential support a countable set of points $\{z_n\}$ (i.e., $\mu(E \setminus \{z_n\}_1^\infty) = 0$). It turns out that any measure for which conditions (3) and (4) are satisfied must be concentrated on sufficiently rarefied subsets of E. Denote by $A = \{\mu\}$ the class of all measures on the nowhere dense compact set E for which (3) and (4) are satisfied.

Theorem 3. Every measure $\mu \in A$ is singular with respect to the Hausdorff h-measure defined by the function $h(r)$, if

$$\int_0 \frac{h(r)}{r^2}\, dr < \infty.$$

(For Hausdorff measures see (3).)

I consider it my pleasant duty to express my gratitude to Prof. S. Ya. Khavinson, under whose supervision this work was carried out.

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Received
12 I 1965

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Note: Figure translations are in progress. See original paper for figures.

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