# A. A. ZINGER

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## Abstract

## Full Text

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# ON A PROBLEM OF B. V. GNEDENKO

*(Presented by Academician Yu. V. Linnik on 24 XII 1964)*

**1.** Let $\xi_1, \xi_2, \ldots$ be a sequence of mutually independent random variables and

$$\zeta_n = \frac{1}{B_n}(\xi_1 + \xi_2 + \dots + \xi_n) - A_n, \quad n = 1, 2, \ldots,$$

be a sequence of normalized sums which, for a suitable choice of the normalizing constants ($B_n \to \infty$), has a proper limiting distribution $G(x)$.

Several years ago B. V. Gnedenko posed the problem of characterizing the class of limiting distributions $G(x)$ in the case where, among the distributions of the random variables $\xi_i$ $(i = 1, 2, \ldots)$, only $r$ different ones occur. Denote this class by $\mathfrak{P}_r$. Some special cases of the problem under consideration were studied in ($^2$,$^3$). In particular, in the note of V. M. Zolotarev and V. S. Korolyuk ($^2$) the case $r = 2$ was investigated. It turned out there that the class $\mathfrak{P}_2$ consists of compositions of two stable laws. The case $r > 2$ remained open. The following example shows that for $r > 2$ the class $\mathfrak{P}_r$ is broader than the class of compositions of a finite number of stable distributions.

**Example.** Take $r = 3$. Put $\lambda = \sqrt{2}$ and construct three symmetric distribution functions $F_j(x)$ $(j = 1, 2, 3)$ as follows:

$$F_1(x) = 1 - F_1(-x) = \begin{cases} \frac{1}{2}, & 0 \le x \le 1, \\ 1 - \frac{1}{3}x^{-\lambda}\left[1 + \frac{1}{\sqrt{3}}\cos\left(\log x - \frac{\pi}{6}\right)\right], & x \ge 1; \end{cases}$$

$$F_2(x) = 1 - F_2(-x) = \begin{cases} \frac{1}{2}, & 0 \le x \le 1, \\ 1 - \frac{1}{2}x^{-\lambda}\left[1 - \frac{1}{\sqrt{3}}\sin(\log x)\right], & x \ge 1; \end{cases} \tag{1}$$

$$F_3(x) = 1 - F_3(-x) = \begin{cases} \frac{1}{2}, & 0 \le x \le 1, \\ 1 - x^{-\lambda}\left[1 - \frac{1}{\sqrt{3}}\cos\left(\log x + \frac{\pi}{6}\right)\right], & x \ge 1. \end{cases}$$

Construct the sequence $\{T_n;\ n = 1, 2, \ldots\}$; $T_n$ is defined as the positive solution of the equation

$$b(T_n) = n,$$

where

$$b(T) = e^{\lambda T}\frac{1}{\sqrt{3}}\left[2\sqrt{3} + \cos\left(T - \frac{\pi}{3}\right) + \frac{2}{3}\cos\left(T + \frac{\pi}{3}\right) - \frac{1}{3}\cos T\right].$$

Next put

$$B_n = \exp T_n, \tag{2}$$

$$n_1(n) = \mathrm{E}\left\{B_n^\lambda\left(1 + \frac{1}{\sqrt{3}}\cos\left(T_n - \frac{\pi}{3}\right)\right)\right\}^*,$$

$$n_2(n) = \mathrm{E}\left\{\frac{2}{3}B_n^\lambda\left[1 + \frac{1}{\sqrt{3}}\cos\left(T_n + \frac{\pi}{3}\right)\right]\right\}, \qquad n_3(n) = n - n_1(n) - n_2(n).$$

* Here the symbol E{ } denotes the integer part of the number enclosed in braces.

Now, in constructing the normalized sums $\zeta_n$, we shall be guided by the following rules: in the sum $\zeta_n$, the random summands $\xi_j$ with distribution $F_j(x)$ are taken with $j = 1, 2, 3$, and as normalizing constants we take $B_n$ from (2).

Using the general theorem on convergence of sums of mutually independent random variables (see (1), § 25), it is easy to verify the existence and nondegeneracy of the limiting distribution in the example under consideration and to compute the spectral function of the limiting distribution

$$H_n(x) = \sum_{j=1}^{3} n_j(n)\left[F_j(B_n x) - E(x)\right] \to$$

$$\to -|x|^{-\lambda}\left[1 + \frac{1}{6}\cos\left(\log|x| - \frac{\pi}{6}\right)\right]\operatorname{sign} x. \tag{3}$$

Here $E(x)$ is the improper distribution.

$$D_n(\varepsilon) = 2\varepsilon^2 H_n(\varepsilon) - 2\int_{|x|<\varepsilon} x H_n(x)\,dx = O(\varepsilon^{2-\lambda}).$$

It is immediately clear from (3) that the limiting distribution is not a composition of stable distributions.

2. The limiting distribution $G(x)$, as is known from the general theory (see, for example, (1), § 24), is infinitely divisible. Let $H(x)$ be the spectral function corresponding to it in the representation of the logarithm of the characteristic function by P. Lévy’ s formula.

Take some $q_0 > 0$ and construct the square matrix of order $r$ $\mathcal{H}(x)$ as follows:

$$\mathcal{H}(x) = \left\| \begin{matrix} H(x) & H(q_0 x) & \dots & H(q_0^{r-1} x) \\ H(q_0 x) & H(q_0^2 x) & \dots & H(q_0^r x) \\ \dots & \dots & \dots & \dots \\ H(q_0^{r-1} x) & H(q_0^r x) & \dots & H(q_0^{2r-2} x) \end{matrix} \right\|. \tag{4}$$

We shall call the limiting distribution $G(x)$ **nondegenerate** if, for at least one pair of positive numbers $x_0, q_0$, the matrices $\mathcal{H}(\pm x_0)$ are nonsingular.*

This note contains results characterizing the class $\mathfrak{P}_r^1$ of limiting distributions for B. V. Gnedenko' s problem in the nondegenerate case. In particular, conditions are established under which the limiting distribution is a composition of stable distributions. Namely, in the nondegenerate case the following theorem holds:

**Theorem 1.** If the limiting distribution $G(x) \in \mathfrak{P}_r^1$, then:

1°. For any $q > 0$, on each of the half-axes of definition $(-\infty, 0)$, $(0, \infty)$, $\mathcal{H}(x)$ satisfies the equation

$$\mathcal{H}(qx) = \mathcal{H}(x) h^{\log q}, \tag{5}$$

where the matrix $h$ does not depend on $x$.

2°. For $x > 0$ (respectively, for $x < 0$) there may be specified constant matrices $A$ and $B$ ($AB = \mathcal{H}(x_0)$) such that $\mathcal{H}(x)B^{-1}$ and $A^{-1}\mathcal{H}(x)$ preserve sign and are monotone.

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* In note (2) it is shown that the spectral function $H(x)$ must satisfy the equation $\beta_0 H(x) + \dots + \beta_{rH}(q^r x) = 0$. The definition of nondegeneracy corresponds to the condition $\beta_0 \neq 0$, $\beta_r \neq 0$.

3°. All characteristic numbers of the matrix $h$ are simple (they do not depend on $q$). They may be written in the form

$$\exp(-\lambda_1); \quad \exp(-\lambda_1 \pm i\nu_{11}); \ \dots \ \exp(-\lambda_1 \pm i\nu_{1k_1}),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\exp(-\lambda_\rho); \quad \exp(-\lambda_\rho \pm i\nu_{\rho 1}); \ \dots \ \exp(-\lambda_\rho \pm i\nu_{\rho k_\rho});$$

here

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_\rho < 2; \quad 0 < \nu_{\alpha\beta} \leq \pi; \quad \sum_{l=1}^{\rho}(1 + 2k_l) = r.$$

The theorem formulated above makes it possible to describe the class $\mathscr{P}_r^1$ in the following way.

Let us denote by $\mathscr{G}_l$ the totality of those distribution laws from Khintchine' s class $L$ (see ($^1$), § 29) whose spectral functions on each of the half-axes $(-\infty, 0)$, $(0, \infty)$ are represented in the form

$$H(x) = |x|^{-\lambda_l} \left\{ a_l + \sum_{j=1}^{k_l} \left[ a_{lj} \cos(\nu_{lj} \log|x|) + b_{lj} \sin(\nu_{lj} \log|x|) \right] \right\},$$

where $a_l \neq 0$; $a_{lj}^2 + b_{lj}^2 \neq 0$; $j = 1, 2, \dots, k_l$.

For $G(x)$ there is the representation

$$G(x) = G_1 * G_2 * \dots * G_\rho(x), \tag{6}$$

where $G_l(x) \in \mathscr{G}_l$; $l = 1, 2, \dots, \rho$.

It should be noted that laws of the form (6) belong to a family of laws of a more general form, studied for the first time by Yu. V. Linnik in work ($^4$) in connection with the investigation of identically distributed linear statistics in repeated homogeneous samples. For the case where the limiting distribution must be a composition of stable ones, the following theorem holds:

**Theorem 2.** In order that the limiting distribution $G(x) \in \mathscr{P}_r^1$ be a composition of stable distributions, it is necessary and sufficient that all characteristic numbers of the matrix $h$ be real and positive.

In this case the eigenvalues of the matrix $h$ may be rewritten in the form $\exp(-\lambda_1)$; $\exp(-\lambda_2)$; ...; $\exp(-\lambda_r)$, where $0 < \lambda_1 < \cdots < \lambda_r < 2$, and $\lambda_l$ $(l = 1, \dots, r)$ are the exponents of stable laws, whose composition forms the limiting law $G(x)$.

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*Note: Figure translations are in progress. See original paper for figures.*

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