MATHEMATICAL PHYSICS

V. P. GACHOK

1965

SovietRxiv

Abstract

Full Text

MATHEMATICAL PHYSICS

V. P. GACHOK

ON THE MOMENT PROBLEM IN QUANTUM FIELD THEORY

(Presented by Academician N. N. Bogolyubov, 27 III 1965)

One of the little-studied relations in field theory is the condition of positive definiteness. This condition is written in the form of an infinite system of inequalities ([1])

$$\sum_{m,n=0}^{N} \bar{\alpha}_n W_{m+n}(f_m \bar{f}_n)\alpha_m \geqslant 0, \qquad ( )$$

where W_n are the Wightman functionals and α_n are complex numbers. In consequence of this condition the functionals $W_n(f_1)$ are moments of a certain positive measure $\sigma(\lambda; f_1)$,

$$W_n(f_1) = \int_{-\infty}^{\infty} \lambda^n \, d\sigma(\lambda; f_1), \qquad f_1(x) \in S_4. \qquad ( )$$

The problem arises of the uniqueness of such a representation. It turns out that this question is closely connected with the self-adjointness of the closure of the field operator $A(f_1)$.

Recently the self-adjointness properties of field operators have been discussed in a number of works ([2–4]) in connection with the use of von Neumann operator rings in field theory ([2–8]). Of these studies we note first of all the work of Borchers and Zimmermann ([4]), in which, with the aid of a theorem of Nelson ([9]), the self-adjointness of the closure of the field operator is proved. In doing so an additional assumption is introduced on the analyticity of the vacuum state ψ_0, which is equivalent to the assumption

$$\left|(\psi_0, [A(f_1)]^n \psi_0)\right| = |W_n(f_1)| < A^n n!, \qquad 0 < A < \infty.$$

In the present paper a scheme is proposed in which the field operators admit self-adjoint closures. The essential condition used here is

$$|W_n(f_1)| < A^n n^{n+B\ln(n+C)}, \qquad 0 \leqslant B < \infty, \quad 0 < C < 1, \qquad ( )$$

and the postulates of the theory, such as Lorentz invariance, spectrality, and causality, are not used.

For the proof we use the idea of the work of Kostyuchenko and Mityagin ([10,11]), based on the use of the condition of positive definiteness and the theory of rigged Hilbert spaces. The scheme for constructing rigged Hilbert spaces in field theory was substantiated in works ([12–14]).

As was already noted in ([15]), the question of the self-adjointness of the closure of the field operator (in ([15]) it corresponds to the shift operator) is closely connected with the problem of uniqueness of the solution of the moment problem. According to Theorem 2, in our scheme the moment problem admits a unique solution.

From the remarks in ([8,16]) it follows that the conditions of the proposed scheme cannot be weakened even to the estimate

$$|W_n(f_1)| < Ann^{(1+\varepsilon)}, \qquad \varepsilon > 0.$$

1. Consider the sequence of spaces ([7,12]) $S_0, S_4, \dots, \dots, S_{4n}, \dots$, where S_0 denotes the space of complex numbers, and S_{4n} denotes the space of complex-valued functions of $4n$ variables, infinitely differentiable and decreasing at infinity, together with all their derivatives, faster than any power of $1/|x|$. With the aid of this sequence of spaces one constructs the ring Σ of sequences $f = \{f_n(x, \dots, x_n)\}$ with multiplication

$$f \times g = \{\dots, (f \times g)_n, \dots\}, \qquad (f \times g)_n = \sum_{i=0}^{n} f_i(x_1, \dots, x_i) g_{n-i}(x_{i+1}, \dots, x_{n-i})$$

and involution $f^* = \{f_n(\overline{x_1, \dots, x_n})\}$.

Let a countable system of norms be given in Σ,

$$\|f\|_p = \sum_{n=0}^{\infty} \sup_{\substack{x \\ \alpha \leq p}} \left| f_n^{(\alpha)}(x_1, \dots, x_n) \right| a_{np}, \qquad p = 1, 2, \dots, \qquad (1)$$

where

$$f_n^{(\alpha)}(x_1, \dots, x_n) = \sum_{0 < \alpha' \leq \alpha} \left| D^{\alpha'} f_n(x_1, \dots, x_n) \right|;$$

$D^{\alpha'}$ is a differentiation operator. In what follows we restrict ourselves to the case when $a_{np} = (np)^{n+B\ln(n+C)}$; $0 \le B < \infty$, $0 < C < 1$. We shall denote this set of elements of Σ by $\Sigma(a_{np})$.

Using ([17,18]) one can show that:

1) the multiplication operation is continuous in $\Sigma(a_{np})$;
2) $\Sigma(a_{np})$ is a nuclear space;
3) the space conjugate to $\Sigma(a_{np})$ is the space of sequences of functionals $\{W_n\}$, $W_n \in S'_{4n}$, with the topology defined by the countable system of norms

$$|W|_p = \sup_n \{|W_n(f_n)|(np)^{-n-B\ln(n+C)}\},$$

and, moreover,

$$\Sigma'(a_{np}) = \bigcup_{p=1}^{\infty} \Sigma'_p(a_{np}),$$

where $\Sigma'_p(a_{np})$ is the space conjugate to $\Sigma_p(a_{np})$, S'_{4n} is conjugate to S_{4n}, and

$$W(f) = \sum_{n=0}^{\infty} W_n(f_n).$$

In what follows, of the functional $W(f)$ we shall require only that it satisfy conditions (α) and (γ). Condition (γ) ensures continuity of the functional $W(f)$. Now we can construct Ω, $V = \Sigma(a_{np})/\Omega$, and the Hilbert space H (in the notation of ([11,12])) with scalar product

$$(\psi(f), \psi(g)) = W(f^* \times g), \qquad \psi(f) \in V. \tag{2}$$

The topology of the nuclear space V is given by means of the system of norms

$$|\psi(f)| = \sum_{n=0}^{\infty} \sup_{\substack{x \\ \alpha \le p \\ f^{(0)} \in \Omega}} \left| f_n^{(\alpha)}(x_1, \dots, x_n) + f_n^{(0)(\alpha)}(x_1, \dots, x_n) \right| a_{np}.$$

2. Introduce the operator of multiplication of elements of $\Sigma(a_{np})$ by the element $\tilde{f}_1 = \{0, f_1(x), 0, \dots\}$, where $f_1(x)$ is a real function from S_4. If

denote this operator by $A(f_1)$, then

$$A(f_1)f = f_1 \times f = \{0,\ f_1(x)f_0, \dots,\ f_1(x)f_n(x_1, \dots, x_n), \dots\}.$$

The operator $A(f_1)$, like any symmetric operator, has a closed symmetric extension, namely $A^{**}(f_1)$, which we shall denote by $\overline{A(f_1)}$ and call the closure of $A(f_1)$. Since this operator is real, it has equal deficiency indices (19). In the case when the deficiency indices are zero $(0, 0)$, the operator $\overline{A(f_1)}$ is self-adjoint. In other cases, by Neumann' s theorem, it admits self-adjoint extensions.

Theorem 1. *The closure of the operator $A(f_1)$ in H, generated by a positive definite functional $W(f)$, is a self-adjoint operator.*

The proof of this theorem reduces to the question of uniqueness of the Cauchy problem in the adjoint space $\Sigma'(a_{np})$. For this purpose, with the aid of the operator $A(f_1)$ one introduces the adjoint operator $A^*(f_1)$, mapping $\Sigma'(a_{np})$ into itself, by the formula

$$\langle A^*(f_1)W_f, g\rangle = \langle W_f, A(f_1)g\rangle = (f, A(f_1)g),$$

where $f, g \in \Sigma(a_{np})$, $W_f \in \Sigma'(a_{np})$, and $\langle\, ,\, \rangle$ denotes the action of the functional. Since $\Sigma(a_{np}) \subset H \subset \Sigma'(a_{np})$, the operator $A^*(f_1)$ is also defined on $\Sigma(a_{np})$. Moreover, here it coincides with $A(f_1)$. Consequently, we may regard $A^*(f_1)$ as an extension of $A(f_1)$ to $\Sigma'(a_{np})$. Consider the equation

$$\frac{d\xi(t)}{dt} = -iA^*(f_1)\xi(t),$$

where $\xi(t)$ is an unknown element in $\Sigma'(a_{np})$.

The Cauchy problem in $\Sigma'(a_{np})$ consists in finding a solution of this equation with initial condition $\xi(0) = f \in \Sigma(a_{np})$. According to Holmgren's principle (20), the solution of this problem is closely connected with the analogous problem in $\Sigma(a_{np})$. Therefore it is enough for us to prove that the series $\exp(it\overline{A(f_1)})f$ has a nonzero radius of convergence. Indeed, by definition,

$$[\overline{A(f_1)}]^k f = \left\{0, \dots, 0, \prod_{j=1}^{k} f_1(y_j)f_0, \dots, \prod_{j=1}^{k} f_1(y_j)f_n(x_1, \dots, x_n), \dots\right\}.$$

Let $q > p$. Since $\Sigma(a_{np}) = \bigcap_{p=1}^{\infty} \Sigma_p(a_{np})$, we estimate the p-th norm of the element $[\overline{A(f_1)}]^k f$ for fixed k:

$$\|[\overline{A(f_1)}]^k f\|_p = \sum_{n=0}^{\infty} \sup_{\substack{x \\ a<p}} \left|([\overline{A(f_1)}]^k f)^{(a)}_{n-k}(x_1, \dots, x_{n-k})\right| a_{np} \leq$$

$$\leq C_1^k \|f\|_q \max_n I(n); \qquad G_1 = \max_x |f_1(x)| > 0,$$

$$I(n) = I_1(n) I_2(n), \qquad I_1(n) = (pn)^n [q(n-k)]^{k-n},$$

$$I_2(n) = (pn)^{B \ln(n+C)} [q(n-k)]^{-B \ln(n-k+C)}.$$

Regarding n as a continuous parameter, one can find $\max_n I(n)$.

Let $E_\lambda^{(1)}(f_1)$ and $E_\lambda^{(2)}(f_1)$ be the spectral families of two different self-adjoint extensions of the operator $A(f_1)$. Then, by virtue of the uniqueness of the Cauchy problem,

$$\int e^{it\lambda}\, d(E_\lambda^{(1)}(f_1) f, g) = \int e^{it\lambda}\, d(E_\lambda^{(2)}(f_1) f, g)$$

for any $f, g \in \Sigma(a_{np})$. Hence $E_\lambda^{(1)}(f_1) = E_\lambda^{(2)}(f_1)$. For V the proof is analogous.

3. We can now prove a theorem expressing the criterion for the representability of the sequence $W_n(f_n) = W_n(f_1)$, where $f_n = \prod_{j=1}^n f_1(x_j)$, in the form (β).

Theorem 2. *In order that the sequence $\{W_n(f_1)\}$ admit the representation (β) with a unique $\sigma(\lambda; f_1)$-measure, it is sufficient that conditions $(\alpha)_\infty$ and (γ) be satisfied.*

Since from the conditions of the theorem it follows that

$$\overline{A(f_1)} = \int_{-\infty}^{\infty} \lambda\, dE(\lambda; f_1),$$

we can compute

$$W_n(f_1) = ([\overline{A(f_1)}]^n \psi_0, \psi_0) = \int_{-\infty}^{\infty} \lambda^n d(E(\lambda; f_1)\psi_0, \psi_0) = \int_{-\infty}^{\infty} \lambda^n d\sigma(\lambda; f_1),$$

where the measure $\sigma(\lambda; f_1)$ is unique and ψ_0 is the vacuum vector. Necessity can be investigated only partially.

In conclusion I express my gratitude to I. Todorov for a valuable discussion, and to M. L. Gorbachuk and N. N. Chaus for valuable discussion of the work ([10]).

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

Received
19 III 1965

REFERENCES

[1] A. S. Wightman, *Problèmes mathématiques de la théorie quantique des champs*, 1959.
[2] H. Araki, J. Math. Phys., **5**, 1 (1964).
[3] A. S. Wightman, Lectures on theoretical physics, Trieste, 1962.
[4] H. J. Borchers, W. Zimmermann, Nuovo Cim., **31**, 1047 (1963).
[5] R. Haag, B. Schroer, J. Math. Phys., **3**, 248 (1962).
[6] R. Haag, D. Kastler, J. Math. Phys., **5**, 848 (1964).
[7] H. J. Borchers, Nuovo Cim., **24**, 214 (1962).
[8] R. F. Streater, J. Math. Phys., **5**, 581 (1964).
[9] F. Nelson, Ann. Math., **70**, 572 (1959).
[10] A. T. Kostyuchenko, B. S. Mityagin, Tr. Mosc. Math. Soc., **9**, 283 (1960).
[11] V. P. Gachok, Ukr. Math. J., **17**, No. 4 (1965).
[12] V. P. Gachok, M. L. Gorbachuk, Ukr. Math. J., **16**, 528 (1964).
[13] K. Maurin, Bull. Acad. Polon. Sci., ser. math., astr., phys., **3**, 115 (1963).
[14] K. Maurin, Bull. Acad. Polon. Sci., ser. math., astr., phys., **3**, 120 (1963).
[15] M. G. Krein, M. A. Krasnosel' skii, UMN, **2**, 60 (1950).
[16] H. Hamburger, Math. Zs., **4**, 186 (1919).
[17] I. M. Gel' fand, G. E. Shilov, *Generalized Functions*, Vol. 2, Moscow, 1958.
[18] I. M. Gel' fand, N. Ya. Vilenkin, *Generalized Functions*, Vol. 4, Moscow, 1961.
F. Riesz, B. Sz.-Nagy, *Lectures on Functional Analysis*, IL, 1954.
[20] I. M. Gel' fand, G. E. Shilov, *Some Questions in the Theory of Differential Equations*, Vol. 3, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.