MATHEMATICS

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Abstract

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ON THE TOPOLOGY OF CONTINUA

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The relative distance $\tilde{\rho}_X(x_1, x_2)$ between points x_1 and x_2 of a continuum X, introduced by C. Mazurkiewicz (^{4}), is the lower bound of the diameters of those of its connected subsets which contain both these points. The relative distance turns X into a new metric space—the **space of the relative metric** of the continuum X. This space (denoted below by $\widetilde{X}$) was investigated by P. S. Urysohn (^{5}). To him also belongs the formulation of the following three problems:

Problem γ. Does there exist a continuum X such that the space $\widetilde{X}$ is zero-dimensional?

Problem δ. Does there exist a continuum X such that the space $\widetilde{X}$, while not being zero-dimensional, contains no connected subsets other than one-point sets?

Problem ε. Does there exist a continuum X such that the space $\widetilde{X}$ is cohesive, while not being connected?

The first of these problems received a positive solution in the author' s paper (^{2}), and the third—in a paper of M. A. Shtan' ko (^{6}).

Below a one-dimensional continuum Δ is constructed, realizing a positive solution of problem δ.

1. Let us call a continuum X **degenerate** if $w\widetilde{X} = \aleph_0{}^*$, $\dim \widetilde{X} = 0$. The basis of the induction process leading to the continuum Δ is the existence, established in (^{2}), of degenerate continua. The degenerate continuum C constructed in (^{2}) is snakelike and, consequently, homeomorphic to a part of the plane (see (^{1})). Therefore:

In any triangle $[x_1, x_2, x_3]$ there can be found a degenerate continuum containing its vertices x_1 and x_2.

2. Consider the regular tetrahedron

$$\Delta_0 = [a, b, c, p],$$

the length of each of whose edges is equal to unity. At the first step of the induction there will be constructed a continuum Δ_1 lying in this tetrahedron. For its construction, choose first of all on the segment $[a, p]$ a sequence of points $a_1, a_2, \ldots, a_m, \ldots$, determined by the equalities $\rho(a_m, a) = 2^{-m}$. In an analogous way the points b_m and c_m $(m = 1, 2, \ldots)$, lying respectively on the segments $[b, p]$ and $[c, p]$, are chosen.

The continuum Δ_1 will be defined as the sum of two sets: a degenerate continuum Γ and a compactum Σ_0.

For the construction of the first of these, consider the triangle $\tau_c = [a, b, p]$. The point of intersection of the segment $[a_{m+1}, b_m]$ with the perpendicular dropped from the point a_m to the segment $[a_{m+1}, b_{m+1}]$ will be denoted by q_m. According to point 1, there exists a degenerate continuum A_m, lying in

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* By $w\widetilde{X}$ is denoted the weight of the space $\widetilde{X}$.

triangle $[a_m, a_{m+1}, q_m]$ and containing the points a_m and a_{m+1}. We set

$$A = (a) \cup \left(\bigcup_{m=1}^{\infty} A_m \right).$$

By B we denote the continuum symmetric to the continuum A with respect to the line joining the point p to the midpoint of the segment $[a, b]$. It can be shown that A and B are expressible continua and that

$$\lim_{m\to\infty} \tilde{\rho}_A(a_m, a) = 0, \qquad \lim_{m\to\infty} \tilde{\rho}_B(b_m, b) = 0. \tag{1}$$

Let Φ be any expressible continuum lying in the triangle $\tau_p = [a, b, c]$ and containing its vertices a and b. By definition,

$$\Gamma = A \cup \Phi \cup B.$$

It is easy to show that Γ is an expressible continuum.

We proceed to the construction of the compactum Σ_0. To this end, having fixed a positive integer m, we project the continuum Φ from the point p onto the triangle $[a_m, b_m, c_m]$. The resulting continuum Φ_m, obviously, contains the points a_m and b_m. And since the described projection is a 2^{-m+1}-shift, we have

$$\Phi_m \subset O(\Phi, 2^{-m+1}). \tag{2}$$

There exists a finite system $T_m^1, T_m^2, \ldots, T_m^{k(m)}$ of regular triangles lying in the triangle $[a_m, b_m, c_m]$, $T_m^k = [a_m^k, b_m^k, c_m^k]$, of diameter less than $2^{-(m+3)}$, such that:

$$(\alpha_1) \quad a_m^1 = a_m, \quad b_m^{k(m)} = b_m.$$

$$(\alpha_2) \quad T_m^k \cap T_m^{k'} = \begin{cases} a_m^{k'} = b_m^k, & \text{if } k' = k+1, \\ \Lambda, & \text{if } |k'-k| > 1. \end{cases}$$

$$(\alpha_3) \quad \text{The set } Z_m = \bigcup_{k=1}^{k(m)} T_m^k \text{ lies in } O(\Phi_m, 2^{-(m+3)}).$$

Each triangle $T_m^k = [a_m^k, b_m^k, c_m^k]$ is the base of two regular tetrahedra lying in Δ_0. Let $(\Delta_0)_m^k$ be the one of them whose fourth vertex (denote it by p_m^k) is separated from the point p by the plane of the triangle $[a_m, b_m, c_m]$:

$$(\Delta_0)_m^k = [a_m^k, b_m^k, c_m^k, p_m^k].$$

From (α_2) it follows that the set

$$(\Sigma_0)_m = \bigcup_{k=1}^{k(m)} (\Delta_0)_m^k$$

is a continuum, and moreover, in accordance with (2) and (α_3), the inclusion $(\Sigma_0)_m \subset O(\Phi, 2^{-m+2})$ holds. Thus,

$$\Sigma_0 = \Phi \cup \left(\bigcup_{m=1}^{\infty} (\Sigma_0)_m \right)$$

turns out to be a compactum. The components of this compactum are, first, all the sets $(\Sigma_0)_m$, and second, the set Φ. We set

$$\Delta_1 = \Gamma \cup \Sigma_0.$$

It follows from (α_1) that Δ_1 is a continuum. The first step in the construction of the continuum Δ is complete.

3. To continue the induction process it is convenient to use the system of mappings

$$f_m^k : \Delta_0 \to (\Delta_0)_m^k \quad (k = 1, 2, \ldots, k(m);\ m = 1, 2, \ldots),$$

each of which is a linear homeomorphism taking the vertices a, b, c, p of the tetrahedron Δ_0, respectively, to the vertices $a_m^k, b_m^k, c_m^k, p_m^k$ of the tetrahedron $(\Delta_0)_m^k$.

Let E be an arbitrary subset of the tetrahedron Δ_0. We put

$$\varphi E = \Gamma \cup \left(\bigcup_{m=1}^{\infty} \bigcup_{k=1}^{k(m)} f_m^k E \right)$$

(thus, in particular, $\Delta_1 = \varphi\Delta_0$). If E is a continuum containing the points a and b, then φE is also a continuum containing the points a and b. Moreover, it is clear that from the inclusion $E' \subseteq E$ there follows the inclusion $\varphi E' \subseteq \varphi E$. Therefore, putting

$$\Delta_{n+1} = \varphi\Delta_n \qquad (n = 0, 1, 2, ...),$$

we obtain a decreasing sequence of continua, each of which contains the points a and b. By definition,

$$\Delta = \bigcap_{n=0}^{\infty} \Delta_n.$$

It can be shown that $\varphi\Delta = \Delta$, and, consequently, Δ is representable in the form of the sum

$$\Delta = \Gamma \cup \left(\bigcup_{m=1}^{\infty} \bigcup_{k=1}^{k(m)} \Delta_m^k \right),$$

in which $\Delta_m^k = f_m^k\Delta$ are continua homeomorphic to Δ.

We describe a plan by following which one can establish that the continuum Δ indeed realizes a positive solution of problem δ.

4. Let G and H be open subsets of the space $\widetilde{\Delta}$ such that $a \in G$, $b \in H$, and $G \cup H = \widetilde{\Delta}$. Using relations (1), one can establish the existence of such indices m and k ($k \leq k(m)$) that $f_m^k a \in G$, $f_m^k b \in H$. Hence, by induction, it follows that the points a and b cannot be separated in the space $\widetilde{\Delta}$ by the empty set, and, thus,

The small inductive dimension of the space $\widetilde{\Delta}$ is positive.

5. **Definition** (see (3)). A continuum Y is called **correctly situated** in the continuum X containing it if the identity mapping $\xi : Y \to X$ is a homeomorphism.

It turns out that both Γ and all the Δ_m^k are correctly situated in Δ. In other words, the following (topologically understood) equality holds:

$$\widetilde{\Delta} = \tilde{\Gamma} \cup \left(\bigcup_{m=1}^{\infty} \bigcup_{k=1}^{k(m)} \widetilde{\Delta}_m^k \right). \tag{3}$$

Notation. The component of a metric space X containing a point $x \in X$ will be denoted by $\varkappa_x X$.

Since $\operatorname{ind} \tilde{\Gamma} = 0$, relation (3) makes it possible to establish that $\varkappa_a \widetilde{\Delta} = a$, $\varkappa_b \widetilde{\Delta} = b$. These equalities, together with the same relation (3), lead, in turn, to the following assertion:

Let x be an arbitrary point of $\widetilde{\Delta}$. If there exist indices m and k ($k \leq k(m)$) such that $x \in \widetilde{\Delta}_m^k$, then $\varkappa_x \widetilde{\Delta} = \varkappa_x \widetilde{\Delta}_m^k$; otherwise $\varkappa_x \widetilde{\Delta} = x$.

It follows from this assertion that:

The space $\hat{\Delta}$ contains no connected subsets except one-point ones.

6. Let

$$\Gamma_{m_1 m_2 \ldots m_n}^{k_1 k_2 \ldots k_n} = \left(f_{m_1}^{k_1} \cdot f_{m_2}^{k_2} \cdots f_{m_n}^{k_n} \right) \Gamma,$$

$$\Delta_{m_1 m_2 \ldots m_n}^{k_1 k_2 \ldots k_n} = \left(f_{m_1}^{k_1} \cdot f_{m_2}^{k_2} \cdots f_{m_n}^{k_n} \right) \Delta.$$

Denote by P the sum of all sets of the form $\Gamma_{m_1 m_2 \ldots m_n}^{k_1 k_2 \ldots k_n}$ (including Γ), and put $Q = \Delta \setminus P$. It is easy to see that a point x^0 of Δ belongs to Q if and only if, for every integer n, there exists a set containing it of the form $\Delta_{m_1 m_2 \ldots m_n}^{k_1 k_2 \ldots k_n}$. What has been said allows us to establish that:

(a) The continuum Δ is locally connected at each of its points belonging to Q.

(b) The set Q, considered in the metric of the continuum Δ, is zero-dimensional.

From (a) it follows that

(c) The identity mapping $\xi : \tilde{\Delta} \to \Delta$ is a homeomorphism on the set Q.

Let the sets P and Q, considered in the metric of the space $\tilde{\Delta}$, be denoted respectively by $P_{\tilde{\Delta}}$ and $Q_{\tilde{\Delta}}$. From (b) and (c) it follows that

(d) $w(Q_{\tilde{\Delta}}) = \aleph_0; \quad \operatorname{ind}(Q_{\tilde{\Delta}}) = 0.$

Now observe that the weight of each of the spaces $\tilde{\Gamma}_{m_1 m_2 \ldots m_n}^{k_1 k_2 \ldots k_n}$ is countable and that they are all closed in $\tilde{\Delta}$.* Therefore

(e) $w(P_{\tilde{\Delta}}) = \aleph_0; \quad \operatorname{ind}(P_{\tilde{\Delta}}) = 0.$

From (d) and (e) we conclude that $w\tilde{\Delta} = \aleph_0$ and that $\operatorname{ind}\tilde{\Delta} \leq 1$. And since, according to item 4, $\operatorname{ind}\tilde{\Delta} > 0$, it follows that

The space $\tilde{\Delta}$ is one-dimensional and has countable weight.

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CITED LITERATURE

[1] R. H. Bing, Duke Math. J., **18**, 653 (1951).
[2] O. V. Lokutsievskii, DAN, **151**, No. 4 (1963).
[3] O. V. Lokutsievskii, DAN, **157**, No. 5 (1964).
[4] S. Mazurkiewicz, Fund. Math., **1** (1920).
[5] P. S. Urysohn, *Works on Topology and Other Fields of Mathematics*, **2**, 1951, p. 517.
[6] M. A. Shtan' ko, UMN, **18**, issue 5 (113) (1963).

* The latter follows from the compactness of $\Gamma^{k_1 k_2 \ldots k_n}_{m_1 m_2 \ldots m_n}$ and from the continuity of the identity mapping $\xi : \tilde{\Delta} \to \Delta$.

Note: Figure translations are in progress. See original paper for figures.