MATHEMATICS

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1965

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Abstract

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CUBATURE FORMULAS WITH A REGULAR BOUNDARY LAYER

In one of the preceding notes (^{1}), error functionals of cubature formulas of the form

$$l(x) = \mathcal{E}_\Omega(x) - \sum_{j=1}^{N} C_j \delta(x - x^{(j)}) = \sum_\gamma l_\gamma \left(\frac{x}{h} - \gamma\right), \tag{1}$$

were considered, where $l_\gamma(x)$ are functionals with bounded support and bounded norm in $L_2^{(m)}$, orthogonal to all polynomials of degree m. Such $l_\gamma(x)$, satisfying conditions (10), (11), and (12) of (^{1}), will be called functionals from $\mathfrak{K}(A, L, m)$. Let H be the matrix of periods of some lattice with determinant equal to one, $|H| = 1$. Representation (1) is, obviously, equivalent to another one, namely:

$$l(x) = \sum_\gamma l_\gamma \left(\frac{x}{h} - H\gamma\right). \tag{2}$$

We shall consider formulas for which:

a) The nodes of all $l_\gamma(x)$ are situated at the points $hH\gamma'$

$$l_\gamma(x) = \mathcal{E}_\gamma(x) - \sum_{\gamma'} C_{\gamma'} \delta(x - hH\gamma'), \qquad \sum \mathcal{E}_\gamma \left(\frac{x}{h} - \gamma\right) = \mathcal{E}_\Omega(x) \tag{3}$$

b) All $l_\gamma(x) \in \mathfrak{K}(L, A, s)$.

(4)

c) All $l_\gamma(x)$ corresponding to interior points coincide: for $d(hH\gamma, \Gamma) > 2Lh$ we have

$$l_\gamma(x) = l_0(x). \tag{5}$$

Such functionals $l(x)$ will be called functionals with a regular boundary layer of order m. The aim of our note is to establish the following main theorem:

Theorem. *As $h \to 0$, for all functionals $l(x)$ with a regular boundary layer of order s, the equality*

$$\|l(x)\|_{L_2^{(m)}} = \left(\frac{h}{2\pi}\right)^m \sqrt{\zeta(H^{-1}, 2m)}\sqrt{|\Omega|} + O(h^{m+1}) \tag{6}$$

holds.

The proof is based on a number of auxiliary lemmas, which we give here.

Lemma 1. *In every functional $l(x)$ with a regular boundary layer, all coefficients C_γ at points whose distance from the boundary is more than $2Lh$ are equal to h^n.*

Indeed:

$$C_\gamma = h^n \sum_{|\gamma'|<L} C^{(0)}_{\gamma-\gamma'}. \tag{7}$$

Since the volume Ω_0 is equal to unity, $(l_0(x), 1) = 0$, we have

$$C_\gamma = h^n, \tag{8}$$

which was required to be proved.

By a **boundary layer** we shall mean the set of nodes $hH\gamma$ in which $C_\gamma \neq h^n$. If, in integration, only points interior with respect to Ω are used, then the boundary layer will be internal. If, for approximating the functional $\mathcal{E}_\Omega(x)$ in $L_2^{(m)}$, exterior points are also used, then we can obtain a two-sided boundary layer, in which only points lying at a distance from the boundary not greater than Lh participate, or else an external boundary layer, in which points lying outside Ω and at a distance from the boundary not greater than $2Lh$ participate. Of course, if the function $\varphi(x) \in L_2^{(m)}(\Omega)$, then only formulas with an internal boundary layer make sense. The number $2L$ will be called the **thickness of the boundary layer**.

Let $m_\gamma(x) = \sum C_{\gamma'}\delta(x - hH\gamma')$. We shall call such functionals **point** functionals. Let, further, $m_\gamma(x) \in \mathfrak{R}(A, L, s)$. Then functionals of the form

$$m(x) = \sum_{hH\gamma \in B_j} m_\gamma(x - hH\gamma), \tag{9}$$

where γ runs through some set B_j, which is a boundary layer of thickness L, will be called a **zero point functional of the boundary layer of order** s. A zero point functional of the boundary layer will be external, internal, or two-sided depending on the location of its support. The thickness of this functional, introduced by analogy with the thickness of the boundary layer, is, generally speaking, equal to $3L$, but it may be smaller. In all that follows it may be regarded as equal to $2L$.

Lemma 2. *The difference of two error functionals $l^{(1)}(x)$ and $l^{(2)}(x)$ with regular boundary layers of orders $s^{(1)}$ and $s^{(2)}$ is a zero point functional of the boundary layer of order*

$$\min\left(s^{(1)}, s^{(2)}\right) - 1. \tag{10}$$

The proof of Lemma 2 is based on an auxiliary lemma.

Lemma 3. *Let $m(x)$ be a finite functional of the form*

$$m(x) = \sum_{|H\gamma|<L} C[\gamma]\delta(x - hH\gamma), \tag{11}$$

orthogonal to all polynomials of degree s, $(m(x), x^{\alpha}) = \sum C[\gamma](hH\gamma)^{\alpha} = 0$ for $|\alpha| \leq s$. The functional $m(x)$ admits the identical representation:

$$m(x) = \sum_{j=1}^{n} \left(M_j(x + hHi_j) - M_j(x)\right), \tag{12}$$

where $(M_j(x), x^{\alpha}) = 0$ for $|\alpha| \leq s - 1$ and $S\{M_j(x)\}$ is concentrated in the smallest parallelepiped with edges parallel to the periods H, containing $S\{m(x)\}$.

The proof of this lemma is carried out by the method of complete induction. We need to establish that the coefficients $C[\gamma]$ admit the representation

$$C[\gamma] = \sum_{j=1}^{n} \hat{\Delta}_j C_j[\gamma], \tag{13}$$

where $\sum C_j[\gamma]\gamma^{\alpha} = 0,\ |\alpha| \leq s - 1;\ \hat{\Delta}_j\varphi[\gamma] = \varphi[\gamma + i_j] - \varphi[\gamma]$.

Let us show that

$$C[\gamma_1, \gamma_2, \dots, \gamma_n] = C[\gamma_1, \gamma_2, \dots, \gamma_{n-1}, \gamma_n{+}1] - C[\gamma_1, \gamma_2, \dots, \gamma_n] + C^{*}[\gamma_1, \gamma_2, \dots, \gamma_{n-1}], \tag{14}$$

where $C^{*}[\gamma_1, \gamma_2, \dots, \gamma_{n-1}]$ is again orthogonal to all polynomials of degree s, while $C[\gamma_1, \gamma_2, \dots, \gamma_n]$ is orthogonal to polynomials of degree $s - 1$, and its support is contained in the smallest parallelepiped containing the support of C_n. Hence Lemma 3 will follow. As $C[\gamma_1, \gamma_2, \dots, \gamma_n]$ and $C^{*}[\gamma_1, \gamma_2, \dots, \gamma_{n-1}]$ it suffices to take the expression

$$C^*[\gamma_1, \gamma_2, \dots, \gamma_{n-1}] = \sum_{\gamma_n'=-L}^{L} C[\gamma_1, \gamma_2, \dots, \gamma_{n-1}, \gamma_n']; \tag{15}$$

$$C_n[\gamma_1, \gamma_2, \dots, \gamma_n] = \begin{cases} 0, & |\gamma_n| \geq L, \\ \displaystyle\sum_{\gamma_n'=-L}^{\gamma_n-1} C[\gamma_1, \gamma_2, \dots, \gamma_{n-1}, \gamma_n'] - (\gamma_n + L)\, C^*(\gamma_1, \gamma_2, \dots, \gamma_{n-1}). \end{cases} \tag{16}$$

Verification of formula (14) and the orthogonality of $C^*[\gamma_1, \gamma_2, \dots, \gamma_{n-1}]$ to polynomials of degree s are obvious. The orthogonality of $C_n[\gamma_1, \gamma_2, \dots, \gamma_n]$ to all polynomials of degree $s-1$ follows from the well-known summation-by-parts formula:

$$\sum_\gamma [\varphi(\gamma+1) - \varphi(\gamma)]\psi(\gamma) = \sum_\gamma \varphi(\gamma)(\psi(\gamma) - \psi(\gamma-1)). \tag{17}$$

It is enough to note that $x_n^{\alpha_n}$ can be represented in the form

$$x_n^{\alpha_n} = \frac{1}{\alpha_n} \hat{\Delta}_n B_{\alpha_n}(x),$$

where B_{α_n} is a Bernoulli polynomial, and to use (16).

Lemma 3 could also be proved in another way—by passing to the Fourier image. Its dual formulation is as follows:

Lemma 3a. *Let Z be the class of rational functions of the form $\Psi(z) = P(z)/z^k$, where $P(z)$ is a polynomial in the variable $z(z_1, \dots, z_n)$, and $z^k = z_1^{k_1} \dots z_n^{k_n}$. Every function $\varphi(z)$ having at the point $I(1, 1, \dots, 1)$ a zero of multiplicity m is representable in the form*

$$\varphi(z) = \sum_{j=1}^{n} (z_j - 1)\varphi_j(z), \tag{18}$$

where $\varphi_j(z)$ are of the same class Z and have at the point I zeros of multiplicity $(m-1)$, and, moreover, the degree of the polynomial $P_j(z)$ in each variable z_j is no higher than the degree of $P(z)$ and $k_j' \leq k$.

Apparently, the proof in the text is no longer than a possible proof of this dual theorem, especially if one takes into account the establishment of their equivalence.

Lemma 3a and Lemma 3 are a particular example of lemmas on the representation of analytic functions having, at a given point $z^{(0)}$, a zero of multiplicity m, in the form

$$\varphi(z) = \sum (z_j - z^{(0)})\varphi_j(z) \tag{19}$$

and of dual lemmas on the corresponding representation of generalized functions $\psi \in K^{(s)}$, where $(\psi(x) * x^\alpha) = 0,\ |\alpha| \leq s$.

For example, L. Schwartz' s theorem on the representation of any generalized function in the form of a differential operator applied to a continuous function is, in essence, also a lemma of the type of Lemma 3.

Corollary. *The sum*

$$\sum_{hH\gamma \in \Omega} m_0(x - hH\gamma) = M_0(x), \tag{20}$$

where $m_0(x)$ are zero point functionals from $\mathfrak{K}(A, L, s+1)$, constitute a functional of a zero boundary layer of order s.

This corollary is obtained immediately if in the left-hand side of (18) one replaces m_0 by its expression on the basis of Lemma 3 and interchanges the order of summation. Lemma 2 follows immediately from Lemma 3.

Corollary of Lemma 2. *Every functional $l(x)$ with a boundary layer of order $s \geqslant m$ can be represented in the form*

$$l(x) = \sum_{hH\gamma \in \Omega} l^* \left(\frac{x}{h} - H\gamma\right) + \sum_{\gamma \in B} l^{**}_\gamma \left(\frac{x}{h} - H\gamma\right), \tag{21}$$

where B is a boundary layer of thickness L, and moreover in such a way that

$$l^* \left(\frac{x}{h} - H\gamma\right) \in \mathfrak{K}(A, L, s), \tag{22}$$

where s_1 is any number greater than s.

It suffices to represent, in the form (9), the difference between our functional and an arbitrary error functional with a regular boundary layer of order s_1. Consider a linear functional $l(x)$ with a regular boundary layer of order $s \geqslant m$. The representation

$$l(x) = 1 - \Phi_0(h^{-1}H^{-1}x) - l^{(1)}(x), \tag{23}$$

is valid, where $l^{(1)}(x)$ is a functional with a regular exterior boundary layer for the domain $\overline{\Omega} = E_n \setminus \Omega$. The extremal function for the functional $l(x)$ has the form

$$u(x) = l(x) * G(x) \tag{24}$$

(see (1)). Using (23), we obtain:

$$u(x) = u_0(x) + C - (l^{(1)}(x) * G(x)), \tag{25}$$

where $u_0(x)$ is an elementary solution of the periodic extremal problem. Let us write explicitly the expression for the norm of the functional $l(x)$:

$$\|l(x)\|^2 = (l(x), u(x)) = (l(x), u_0(x)) - (l(x) * G(x) * l^{(1)}(-x))\big|_{x=0} \tag{26}$$

and replace $l(x)$ and $l^{(1)}(x)$ by their representation (21). We obtain, repeating the estimates given in (1):

$$l(x) * G(x) * l^{(1)}(-x)\big|_{x=0} = O(h^{2m+1}). \tag{27}$$

By direct calculation one can also show that

$$(l(x), u_0(x)) = \frac{h^{2m}}{(2\pi)^{2m}} \zeta(H^{-1}, 2m)|\Omega| + O(h^{2m+1}). \tag{28}$$

Comparing (25), (26), and (27), we obtain the proof of the main theorem.

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Received
3 V 1965

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1. S. L. Sobolev, *Dokl. Akad. Nauk SSSR*, **162**, No. 5 (1965).

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