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Abstract

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ON THE ROOTS OF TRIGONOMETRIC POLYNOMIALS

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In this note some theorems are established on the location and number of roots of even and odd trigonometric polynomials, depending on properties of the roots of the algebraic polynomials associated with them.

Let all roots of the polynomials $Q_1(x) = \sum_0^n a_p^{(1)} x^p$ and $Q_2(x) = \sum_{p=0}^n a_p^{(2)} x^p$, with $a_n^{(1)} \neq 0$ and $a_n^{(2)} \neq 0$, be real and mutually interlacing:

$$x_1^{(1)} < x_1^{(2)} < x_2^{(1)} < \dots < x_n^{(1)} < x_n^{(2)}, \tag{1}$$

and let $\rho_0 = \max\{|x_1^{(1)}|,\ |x_2^{(2)}|\}$.

Form the trigonometric polynomials

$$C_1(\rho, \varphi) = \sum_{p=0}^n a_p^{(1)} \rho^p \cos p\varphi, \qquad C_2(\rho, \varphi) = \sum_{p=0}^n a_p^{(2)} \rho^p \cos p\varphi.$$

Theorem 1. For $\rho \geq \rho_0$, the trigonometric polynomials C_1 and C_2 have on $[0, \pi]$ mutually interlacing real roots (φ),

$$0 \leq \gamma_1^{(2)} < \gamma_1^{(1)} < \gamma_2^{(2)} < \dots < \gamma_n^{(2)} < \gamma_n^{(1)} \leq \pi, \tag{2}$$

$$\lim_{\rho \to \infty} \gamma_k^{(1)}(\rho) = \lim_{\rho \to \infty} \gamma_k^{(2)}(\rho) = (2k-1)\pi/2n. \tag{3}$$

Proof. Without loss of generality, we shall consider $a_n^{(1)} = a_n^{(2)} = 1$. We prove the theorem for $\rho > \rho_0$. Represent the indicated trigonometric polynomials in the form

$$C_1 = \frac{1}{2} \prod_{k=1}^n (\rho e^{\varphi i} - x_k^{(1)}) + \frac{1}{2} \prod_{k=1}^n (\rho e^{-\varphi i} - x_k^{(1)});$$

$$C_2 = \frac{1}{2}\prod_{k=1}^{n}(\rho e^{\varphi i} - x_k^{(2)}) + \frac{1}{2}\prod_{k=1}^{n}(\rho e^{-\varphi i} - x_k^{(2)}).$$

Let

$$\psi_1(\rho,\varphi) = \arg\prod_{k=1}^{n}(\rho e^{\varphi i} - x_k^{(1)}),$$

$$\psi_2(\rho,\varphi) = \arg\prod_{k=1}^{n}(\rho e^{\varphi i} - x_k^{(2)}).$$

Mark on the real axis the roots of the polynomials $Q_1(x)$ and $Q_2(x)$. Connect the points thus obtained with the point $\rho e^{\varphi i}$ on the semicircle; we obtain the angles $(\varphi_k^{(1)})_1^n$ and $(\varphi_k^{(2)})_1^n$, where

$$\varphi_1^{(1)} < \varphi_1^{(2)} < \varphi_2^{(1)} < \varphi_2^{(2)} < \ldots < \varphi_n^{(1)} < \varphi_n^{(2)} \tag{4}$$

for $\varphi \neq 0, \pi$; $\varphi_p^{(1)} = \varphi_p^{(2)} = 0$ $(p = 1, 2, \ldots, n)$ for $\varphi = 0$; $\varphi_p^{(1)} = \varphi_p^{(2)} = \pi$ $(p = 1, 2, \ldots, n)$ for $\varphi = \pi$. It is easy to see that $\psi_1 = \sum_{p=1}^{n}\varphi_p^{(1)}$.

and $\psi_2 = \sum_{p=1}^{n}\varphi_p^{(2)}$. The functions ψ_1 and ψ_2 are continuous and single-valued in φ (for any fixed $\rho > 0$) and, as φ varies from 0 to π, increase monotonically from 0 to $n\pi$ (for $\rho > \rho_0$); moreover, according to (4), $\psi_1 < \psi_2$ if $\varphi \neq 0, \pi$, and $\psi_1 = \psi_2$ if $\varphi = 0, \pi$. It also follows from inequalities (4) that, for $\varphi \neq 0, \pi$,

$$\psi_1 - \varphi_1^{(1)} > \psi_2 - \varphi_n^{(2)} \quad \text{or} \quad \psi_2 < \psi_1 + (\varphi_n^{(2)} - \varphi_1^{(1)}) < \psi_1 + \pi/2.$$

Thus, for $\varphi \neq 0, \pi$,

$$\psi_1 < \psi_2 < \psi_1 + \pi/2. \tag{5}$$

The polynomial $C_1(\rho,\varphi) = |Q(\rho e^{\varphi i})|\cos\psi_1$ (respectively $C_2(\rho,\varphi)$) vanishes on $[0,\pi]$ for those values of the angle φ for which the quantity ψ_1 (respectively ψ_2) is equal to $\pi/2, 3\pi/2, \ldots, (2n-1)\pi/2$. Taking (5) into account, we obtain the assertion of the theorem for $\rho > \rho_0$. In this case, in (2), a strict inequality holds at the endpoints.

If $\rho = \rho_0$ (for definiteness take $|x_1^{(1)}| < x_n^{(2)} = \rho_0$), then $\psi_1(\rho_0, 0) = 0$, $\psi_2(\rho_0, 0) = \pi/2$, and $\gamma_1^{(2)} = 0$. Similarly, when $x_n^{(2)} < |x_1^{(1)}| = \rho_0$, $\gamma_n^{(1)} = \pi$, and when $|x_1^{(1)}| = x_n^{(2)} = \rho_0$, $\gamma_1^{(2)} = 0$, $\gamma_n^{(1)} = \pi$.

The zeros φ of the polynomial $C_1(\rho,\varphi)$ will be found as the roots of the equation

$$\rho^n \cos n\varphi + \sum_{k=0}^{n-1} a_k^{(1)}\rho^k\cos k\varphi = 0. \tag{6}$$

With respect to the variable ρ we have:

$$\cos n\varphi + O(1/\rho) = 0.$$

Thus, as $\rho \to \infty$, the zeros $\left(\gamma_k^{(1)}\right)_{k=1}^{n}$ tend to the quantities $(2k-1)\pi/2n$ ($k = 1, 2, \dots, n$). Relation (3) for the zeros $\left(\gamma_k^{(2)}\right)_{k=1}^{n}$ is proved analogously.

Corollary. If $a_n^{(1)}/a_n^{(2)} > 0$, then for $\rho > \rho_0$ there are $n+1$ intervals on $[0, \pi]$, in each of which the polynomials C_1 and C_2 have the same sign. For $\rho = \rho_0$ the number of such intervals is n, if $|x_1^{(1)}| \neq x_n^{(2)}$, or $n-1$, if $|x_1^{(1)}| = x_n^{(2)}$.

Remark 1. The assertion of the theorem remains valid for $\rho > \rho_0$, if the roots of the polynomial $Q_1(x)$ coincide completely or partially with the roots of the polynomial $Q_2(x)$, with the exception of at least one adjacent pair.

Remark 2. Theorem 1 is also valid in the case when the degrees of the algebraic polynomials differ by one (their roots being mutually separated). In this case, for $\rho \geq \rho_0$ and $0 < \varphi < \pi$, the inequality

$$\psi_1 < \psi_2 < \psi_1 + \pi$$

holds (ψ_1 refers to the polynomial of smaller degree).

Let all the roots of the polynomial

$$Q(x) = \sum_{p=0}^{n} a_p x^p$$

with real coefficients, whether simple or multiple, lie in the disk of radius ρ_0 (including the boundary).

Theorem 2. The trigonometric polynomial

$$C(\rho, \varphi) = \sum_{p=0}^{n} a_p \rho^p \cos p\varphi$$

has on $[0, \pi]$: 1) for $\rho > \rho_0$, n distinct real roots (in φ); 2) for $\rho \in (0, \rho_0]$, the number of real roots counted with multiplicity is greater than or equal to m, where m is the number of roots of the polynomial $Q(x)$ in the disk of radius ρ (including the boundary), counted with their multiplicities.

The proof of the first part is carried out in the same way as in Theorem 1, by constructing the function $\psi(\rho, \varphi)$, taking into account the pairwise conjugacy of the roots and the continuous variation of the angles $(\varphi_k)_0^n$ as φ increases from zero to π.

For the proof of part 2, note that each of the zeros $\{\gamma_j(\rho)\}$ of the polynomial $C(\rho, \varphi)$ is a continuous branch in ρ. As $\rho \to \infty$, each of the branches $\{\gamma_j(\rho)\}$ goes to infinity (see Theorem 1). On the other hand, each one of the branches $\{\gamma_i(\rho)\}$ will pass through the corresponding root of the algebraic polynomial $\sum_{p=0}^{n} a_p x^p$, since $C(\rho, \varphi) = \operatorname{Re} Q(x)$ for any complex x. Thus, the circle of radius $\rho \le \rho_0$ intersects at least once each of the branches that pass through the roots of the algebraic polynomial lying in the disk of radius ρ (including the boundary).

Corollary. If the trigonometric polynomial

$$C(\varphi) = \sum_{k=0}^{n} A_k \cos k\varphi$$

is such that the polynomial

$$Q(x) = \sum_{k=0}^{n} A_k x^k$$

has all its roots inside the unit disk, then $C(\varphi)$ has n distinct real roots on $[0, \pi]$. If, however, the polynomial $Q(x)$ has in the unit disk (including the boundary) only $m < n$ roots, then the number of real roots of $C(\varphi)$ on $[0, \pi]$, counted with multiplicity, is greater than or equal to m.

Example 1.

$$C(\varphi) = \cos 3\varphi + 0.1 \cos 2\varphi - 0.4 \cos \varphi - 0.45.$$

$x^3 + 0.1x^2 - 0.4x - 0.45 = (x - 0.9)(x^2 + x + 0.5)$; $\quad x_1 = 0.9$; $\quad x_{2,3} = -0.5(1 \pm i)$;

$\rho_0 = 0.9$; $C(\varphi)$ has on $[0, \pi]$ the roots

$$\gamma_1 \approx 14^\circ, \quad \gamma_2 \approx 99^\circ.5, \quad \gamma_3 \approx 148^\circ.6.$$

Example 2.

$$C(\varphi) = \cos 3\varphi - 5 \cos 2\varphi + 12 \cos \varphi - 8.$$

$$x^3 - 5x^2 + 12x - 8 = (x - 1)(x^2 - 4x + 8); \quad x_1 = 1; \quad x_{2,3} = 2(1 \pm i);$$

$C(\varphi)$ has on $[0, \pi]$ one root $\gamma = 0$.

Let all roots of the polynomial

$$Q(x) = \sum_{k=0}^{n} a_k x^k$$

be positive and have an upper bound $\rho_0 > 0$.

Theorem 3. All n real roots of the polynomial

$$C[\rho, \varphi] = \sum_{k=0}^{n} a_k \rho^k \cos k\varphi$$

on $(0,\pi)$, as ρ increases (from ρ_0), increase monotonically, tending to their limiting values (see Theorem 1).

Let the polynomials

$$Q(x)=\sum_{p=0}^{n} a_p x^p \quad \text{and} \quad C(\rho,\varphi)=\sum_{p=0}^{n} a_p \rho^p \cos p\varphi$$

satisfy the conditions of Theorem 2.

Theorem 4. In the half-plane $\operatorname{Im} z>0$, the branches $\{\gamma_m(\rho)\}$ $(m=1,2,\dots,n)$, as $\rho\to\infty$, approach rays issuing from the point

$$-\frac{1}{n}\frac{a_{n-1}}{a_n}$$

on the real axis. The angles of inclination are

$$\{(2m-1)\pi/2n\}_{m=1}^{n}.$$

Theorem 5. If $Q_1(x)$ and $Q_2(x)$ satisfy the conditions of Theorem 1, then

$$S_1(\rho,\varphi)=\sum_{p=1}^{n} a_p^{(1)} \rho^p \sin p\varphi, \qquad S_2(\rho,\varphi)=\sum_{p=1}^{n} a_p^{(2)} \rho^p \sin p\varphi$$

for $\rho>\rho_0$ have on $(0,\pi)$ mutually interlacing real roots

$$0<\gamma_1^{(2)}<\gamma_1^{(1)}<\gamma_2^{(2)}<\dots<\gamma_{n-1}^{(2)}<\gamma_{n-1}^{(1)}<\pi,$$

$$\lim_{\rho\to\infty}\gamma_k^{(1)}(\rho)=\lim_{\rho\to\infty}\gamma_k^{(2)}(\rho)=k\pi/n.$$

The proof of this theorem is analogous to the proof of Theorem 1.

Let the polynomial

$$Q(x)=\sum_{k=0}^{n} a_k x^k$$

satisfy the conditions of Theorem 2. We shall call the polynomial

$$Q_1(x)=\sum_{k=1}^{n} a_k x^k + B_0$$

equivalent to the polynomial $Q(x)$, if B_0 is chosen so that the disk in which the roots of $Q_1(x)$ are located

(as before, with center at the origin) has the smallest possible radius ρ_0. Obviously, $\rho_0 \leqslant \bar{\rho}_0$.

Theorem 6. The trigonometric polynomial

$$S(\rho, \varphi) = \sum_{k=1}^{n} a_k \rho^k \sin k\varphi$$

has on $(0, \pi)$: 1) for $\rho > \bar{\rho}_0$, $n-1$ distinct real roots (in φ); 2) for $\rho \leqslant \bar{\rho}_0$, the number of real roots, counted with multiplicity, is greater than or equal to m, where m is the number of real roots, counted with their multiplicities, of the polynomial $Q'(x)$ that are located in the interval $(-\rho, \rho)$.

If by $\{\gamma_j(\rho)\}$ we denote the real roots of the polynomial $S(\rho, \varphi)$, then, taking into account the proof of Theorems 5 and 2, in the present case it is necessary only to show that the branches $\{\gamma_i(\rho)\}$, all of them or some of them, intersect the real axis at the roots of the polynomial $Q'(x)$. Since $S(\rho, \varphi)$ and $S(\rho, \varphi)/\varphi$, as algebraic polynomials in ρ, will have the same roots for any $\varphi \neq 0$, to find the indicated points we seek the roots of the expression

$$\lim_{\varphi \to 0} \sum_{k=1}^{n} a_k \rho^k \sin k\varphi \,/\, \varphi = \rho \sum_{k=1}^{n} a_k k \rho^{k-1},$$

whence the assertion of the theorem follows.

Corollary. If the polynomial $S(\varphi) = \sum_{k=1}^{n} B_k \sin k\varphi$ is such that the polynomial

$$\sum_{k=1}^{n} B_k x^k + B_0$$

has all its roots inside the circle of unit radius, then the polynomial $S(\varphi)$ has on $(0, \pi)$ $n-1$ distinct real roots; if, however, on the interval $(-1; +1)$ the polynomial

$$\sum_{k=1}^{n} B_k k x^{k-1}$$

has $m (< n-1)$ roots (counted with multiplicity), then the number of real roots of the polynomial $S(\varphi)$, counted with multiplicity, is greater than or equal to m.

Example 3. $S(\varphi) = \sin 3\varphi + 0.1 \sin 2\varphi - 0.4 \sin \varphi$. For $a_0 = 0.45$ the polynomial

$$x^3 + 0.1x^2 - 0.4x - 0.45 = (x - 0.9)(x^2 + x + 0.5);$$

$\rho_0 < 1$ (see Example 1); $S(\varphi)$ has on $(0, \pi)$ the roots $\gamma_1 \approx 55^\circ.5$; $\gamma_2 \approx 128^\circ$.

Example 4. $S(\varphi) = 2 \sin 3\varphi - 4.5 \sin 2\varphi - 6 \sin \varphi$.

$$(2x^3 - 4.5x^2 - 6x)' = 6(x + 0.5)(x - 2);$$

$x_1 = -0.5;\ x_2 = 2;\ S(\varphi)$ has on $(0, \pi)$ one root $\gamma \approx 125^\circ.5$.

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Note: Figure translations are in progress. See original paper for figures.