# FREE \(T\)-SUMS OF MULTIOPERATOR FIELDS

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## Abstract

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*MATHEMATICS*

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# FREE $T$-SUMS OF MULTIOPERATOR FIELDS

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The theory, constructed by L. A. Skornyakov (2,3), of free $T$-extensions of linear algebras is transferred in the present note to a broader class of multioperator algebras introduced by A. G. Kurosh (1). Theorem 2 gives a positive solution to the question of the existence of isomorphic continuations for any two free $T$-decompositions of a multioperator, in particular nonassociative, field. For a nonassociative field this question was not considered in (3), although it is natural after Theorem 2 and Theorem § 5 of (3). Theorem 1 generalizes the analogous theorem of (3). The incorrectness of the original proof of this theorem (in (3)) is noted.

Below, a nonempty system of operations $\Omega = \{\omega, \ldots\}$, $n(\omega) \geq 2$, and a field $P$ are assumed fixed.

§ 1. An $\Omega$-algebra $A$ over the field $P$ (1) is called an $\Omega$-algebra with divisions (a multioperator field, or $\Omega$-field) if, for each $\omega \in \Omega$, every equation of the form

$$a_1 \ldots a_{i-1} x a_{i+1} \ldots a_n \omega = c, \qquad 0 \neq a_j,\ c \in A, \tag{*}$$

is solvable (uniquely solvable) in $A$ for $i = 1, 2, \ldots, n = n(\omega)$.

A single-valued mapping $\theta$ of an $\Omega$-algebra $A$ into an $\Omega$-algebra $K$ augmented formally by the symbol $\infty$ is called a $T$-homomorphism (2) if $\theta^{-1}(0)$ is nonempty and the following conditions are satisfied:

1. If $a_i \in A$, $\xi_j \in P$, $\theta(a_i) \neq \infty$, $i = 1, 2, \ldots, n$, then

$$\theta(\xi_1 a_1 + \xi_2 a_2) = \xi_1 \theta(a_1) + \xi_2 \theta(a_2), \quad \theta(a_1 \ldots a_n \omega) = \theta(a_1) \ldots \theta(a_n) \omega, \quad \omega \in \Omega,\ n = n(\omega).$$

2. If $\theta(a_i) \neq 0$, $i = 1, 2, \ldots, n$, and for at least one $1 \leq i_0 \leq n$

$$\theta(a_{i_0}) = \infty,$$

then

$$\theta(a_1 \ldots a_{i_0} \ldots a_n \omega) = \infty.$$

It is immediately verified that the $T$-homomorphic image of an $\Omega$-algebra with divisions is an $\Omega$-algebra with divisions. The successive performance of two $T$-homomorphisms is again a $T$-homomorphism, if, by definition, one sets $\theta(\infty) = \infty$.

§ 2. Let a $T$-homomorphism $\theta$ and an equation $(*)$, not solvable in the $\Omega$-algebra $A$, be fixed. A basis $\Sigma = \Phi \cup \Psi$ of the additive vector space $A^+$ of the $\Omega$-algebra $A$ will be called $(\theta, *)$-regular if: 1) $\Phi$ is a basis of the subspace $E(\Phi) = \{a \in A^+ \mid \theta(a) \neq \infty\}$; 2) for all $a \in E(\Psi) \setminus 0$, $\theta(a) = \infty$; 3) the subspace

$$L = E(\{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n\}),$$

where the $a_j$ are the terms of equation $(*)$, has the basis set

$$\Sigma \cap L = \Sigma'.$$

In the basis $S$, constructed over the set $\Sigma \cup \lambda$ (2), of the free sum

$$A_\lambda = A * F(\lambda)$$

of the $\Omega$-algebra $A$ and the free $\Omega$-algebra $F(\lambda)$ with one free generator $\lambda$, we fix

$$b_1 = a'_1 \dots a'_{i-1} \lambda a'_{i+1} \dots a'_n \omega, \qquad a'_j \in \Sigma',$$
$$E(\{a'_1, \dots, a'_{i-1}, a'_{i+1}, \dots, a'_n\}) = L.$$

The word $b_1$ occurs as a summand in the $S$-canonical representation of the element

$$p = a_1 \dots a_{i-1} \lambda a_{i+1} \dots a_n \omega - c$$

corresponding to equation $(*)$. Then, under the natural epimorphism

$$A_\lambda \to B = A_\lambda / I,$$

where $I$ is the ideal of the $\Omega$-algebra $A_\lambda$ generated by the element $p$, the set

$$S^{b_1} = \{x \in S \mid l_{b_1}(x) = 0\}$$

of elements of $b_1$-length zero (2) is mapped one-to-one onto a basis of the vector space $B^+$ of the $\Omega$-algebra $B$. In the $\Omega$-algebra $B \supset A$, equation $(*)$ is solvable (see (2)). Using

basis $S^{b_1}$, one can carry over to the case of multioperator algebras the proof of Lemma 6 of (2), stating that every $T$-homomorphism of an $\Omega$-algebra $A$ into an $\Omega$-algebra $K$ with divisions can be extended to the obtained extension $B$. Hence, as in (2), we obtain:

**Proposition 1.** Every $\Omega$-algebra $A$ over a field $P$ possesses a free $T$-extension $\mathfrak{A} \supset A$ in the sense of (2), uniquely determined up to an isomorphism identical on $A$. If, moreover, for any $0 \neq x_j \in A$, $\omega \in \Omega$, one has $x_1 \dots x_n \omega \neq 0$, then $\mathfrak{A}$ is an $\Omega$-field.

**Corollary 1.** Every $\Omega$-algebra can be embedded in an $\Omega$-algebra with divisions.

This corollary generalizes B. H. Neumann' s theorem $(^4)$, proved for linear algebras.

Proposition 1 makes it possible to define the concepts (see $(^{2,3})$): free $\Omega$-field, free $T$-sum of $\Omega$-fields. The following propositions are true and are immediate generalizations of theorems from $(^{2,3})$.

**Proposition 2.** Every $\Omega$-field is a $T$-homomorphic image of some free $\Omega$-field.

**Proposition 3.** An $\Omega$-subfield $\mathfrak{B}$ of the free $T$-sum $\sum_T^T \mathfrak{A}_\sigma * \mathfrak{F}$ is decomposable into the free $T$-sum

$$\mathfrak{B} = \sum_T^T (\mathfrak{B} \cap \mathfrak{A}_\sigma) * \mathfrak{F}_1,$$

where $\mathfrak{F}$ and $\mathfrak{F}_1$ are free $\Omega$-fields.

For free $T$-sums of multioperator fields, besides properties analogous to I–V in $(^3)$, the following also hold:

**A1** (validity of Mal' cev' s postulate). If in each of the $\Omega$-fields $\mathfrak{A}_\alpha$ an $\Omega$-subfield $\mathfrak{B}_\alpha$ is taken and $\mathfrak{A} = \sum_T^T \mathfrak{A}_\alpha$, then the $\Omega$-subfield $\mathfrak{B}$ generated by all $\mathfrak{B}_\alpha$ is their free $T$-sum and $\mathfrak{B} \cap \mathfrak{A}_\alpha = \mathfrak{B}_\alpha$.

**A2.** If $M$ is a finite subset of the $\Omega$-field $\mathfrak{A} = \sum_T^T \mathfrak{A}_\alpha$, then there exists a finite set of indices $\alpha_1, \dots, \alpha_n$ such that

$$M \subset \sum_{i=1}^{n\,T} \mathfrak{C}_{\alpha_i},$$

where $\mathfrak{C}_{\alpha_i}$ is an $\Omega$-subfield of the $\Omega$-field $\mathfrak{A}_{\alpha_i}$. Each $\mathfrak{C}_{\alpha_i}$ has, as an $\Omega$-field, a finite number of generators.

§ 3. Let, for the $\Omega$-field $\mathfrak{A}$, there be a free $T$-decomposition:

$$\mathfrak{A} = \sum_T^T \mathfrak{A}_\sigma.$$

Then, by the definition of the free $T$-sum,

$$\Sigma^* \mathfrak{A}_\sigma = A_1 \subset \dots \subset A_\beta \subset \dots \subset A_\theta = \mathfrak{A}.$$

In this transfinite sequence of extensions, for every $0 \neq x \in \mathfrak{A}$ there is uniquely determined a height $h(x)$—such an indefinite ordinal number $\alpha \geqslant 1$ that $x \in$

$A_\alpha \setminus A_{\alpha-1}$ (here $A_0$ is to be understood as the empty set), and a degree $s_\alpha(x)$ ($^2$).

The $\Omega$-subfield generated by a set $Z \subset \mathfrak{A}$ will be denoted by $T(Z)$. Suppose that, as an $\Omega$-field, $\mathfrak{A}$ is generated by some finite subset of it. An ordered set $X = \{x_i \in \mathfrak{A} \mid x_i \neq 0,\ i = 1, \dots, m\}$ will be called a **finite system of generators** (f.s.g.) of the $\Omega$-field $\mathfrak{A}$, if $T(X) = \mathfrak{A}$ and $h(x_i) \leq h(x_{i+1})$, $i = 1, 2, \dots, m-1$. If the subset $X_\alpha = X \cap (A_\alpha \setminus A_{\alpha-1})$ of the f.s.g. $X$ is nonempty, then by $s_\alpha(X)$ we shall denote the sum $\sum_i s_\alpha(x_i)$, $x_i \in X_\alpha$.

On the set $\mathfrak{P}$ of all f.s.g. of the $\Omega$-field $\mathfrak{A}$ we define a quasi-ordering. Let $X = \{x_1, \dots, x_m\}$, $Y = \{y_1, \dots, y_n\} \in \mathfrak{P}$. By definition, we set $X \leq Y$ if one of the following conditions is satisfied: 1) $m < n$, 2) $m = n$, $h(x_i) \leq h(y_i)$, $i = 1, 2, \dots, n$, and at least one of the inequalities is strict; 3) $m = n$, $h(x_i) = h(y_i)$, $s_{\alpha_i}(X) \leq s_{\alpha_i}(Y)$ for all $\alpha_i = h(x_i)$, $i = 1, 2, \dots, n$. It is easily verified that the relation thus defined is reflexive and transitive and therefore is a partial ordering on the set $\mathfrak{P}$ of classes of equivalent f.s.g.: $X \sim Y$ if $m = n$, $h(x_i) = h(y_i)$, $s_{\alpha_i}(X) = s_{\alpha_i}(Y)$ for all $\alpha_i = h(x_i)$, $i = 1, 2, \dots, n$. Since the minimality condition holds for $\mathfrak{P}$, we thus arrive at the definition of **minimal** f.s.g. (m.f.s.g.) of $\Omega$-fields $\mathfrak{A}$. It is clear that

the number of elements in any two m.k.s.o. is the same and does not depend on the $T$-decomposition under consideration.

Denote by $n(X)$ and $r(X)$, respectively, the number of elements of a finite set $X$ of the $\Omega$-group $\mathfrak{A} = \Sigma^T \mathfrak{A}_\sigma$ and of its subset $X \setminus \left(\bigcup_\sigma \mathfrak{A}_\sigma\right)$.

Let, further,

$$\bar{A}_0 = \{x \in A_1 \mid s_1(x) = 0\},$$

where $A_1 = \Sigma^* \mathfrak{A}_\sigma$; $X_0 = X \cap \bar{A}_0$. If $X$ is a k.s.o. of the $\Omega$-group $\mathfrak{A}$ and there exists a finite set $\bar{X}_0 \subset \bar{A}_0$ such that

$$T(\bar{X}_0) = T(X_0), \qquad r(\bar{X}_0) < r(X_0), \qquad n(\bar{X}_0) \leq n(X_0) + 1,$$

then the passage to the k.s.o. $X' = (X \setminus X_0) \cup \bar{X}_0$ will be called an $r$-reduction. Any k.s.o. $X$, in no more than $r(X_0)$ steps, is brought to an $r$-irreducible form that admits no $r$-reductions.

Let $Y$ be an m.k.s.o. of the $\Omega$-group $\mathfrak{A}$, and let $X$ be the $r$-irreducible k.s.o. obtained from $Y$. Denote it by $X = \pi(Y)$ and call it a regular k.s.o. of the $\Omega$-group $\mathfrak{A}$.

As an analogue of Grushko' s theorem, the following has been obtained.

**Proposition 4.** *The elements of a regular k.s.o.* $X = \pi(Y)$ *belong to separate* $T$*-summands.*

The proof of this proposition is based on a specialization of the methods, developed in papers ($^1$,$^3$ ,$^5$ ), for constructing bases in the $\Omega$-subalgebra of a free sum of $\Omega$-algebras.

**Corollary 2.** *An $\Omega$-group with $n$ generators decomposes into a free $T$-sum of no more than $2n$ $T$-indecomposable finitely generated $\Omega$-subgroups.*

**Corollary 3.** *A free $\Omega$-group with $n$ free generators is not decomposable into a free $T$-sum containing more than $n$ summands.*

Hence, using property A2, we arrive at theorem *:

**Theorem 1.** *The cardinality of the set of free generators of a free $\Omega$-group is an invariant of this $\Omega$-group.*

**Theorem 2.** *Any two free $T$-decompositions of an $\Omega$-group have isomorphic refinements.*

To prove Theorem 2, in view of Proposition 3 and properties III and IV for free $T$-sums (see ($^3$)), it suffices to establish an isomorphism of the free $\Omega$-groups $\mathfrak{F}_1$ and $\mathfrak{F}_2$ for any two free $T$-decompositions:

$$\mathfrak{C} *_T \mathfrak{F}_1 = \mathfrak{C} *_T \mathfrak{F}_2,$$

where $\mathfrak{C}$ is an arbitrary $\Omega$-group.

Let $M_1$ and $M_2$ be some fixed systems of free generators of the free $\Omega$-groups $\mathfrak{F}_1$ and $\mathfrak{F}_2$, and suppose

$$\operatorname{card} M_1 < \operatorname{card} M_2, \tag{**}$$

and $M_2$ is an infinite set. Then, using A2, by the usual argument we arrive at a contradiction. If $\mathfrak{C}$ is generated, as an $\Omega$-group, by a finite set and if $M_2$ is finite, then $(**)$ leads to a contradiction with Corollary 2. But when $M_1, M_2$ are finite, by virtue of property A2 one may restrict oneself only to the case considered, that of a finitely generated $\Omega$-group $\mathfrak{C}$. Theorem 2 is proved.

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* In paper (3) this is Theorem § 5, formulated for a free nonassociative ring. The proof given there is unconvincing, since it contains an incorrect reference (p. 309) to paper (5). As for Theorem 5 of (3), which is also affected by the noted incorrectness, the question of the validity of this theorem has not been clarified. Corollary 2 of the present note contains a weaker assertion.

*Note: Figure translations are in progress. See original paper for figures.*

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