A. B. SHABAT

1965

SovietRxiv

Abstract

Full Text

A. B. SHABAT

ON A CERTAIN PROPERTY OF SOLUTIONS OF SECOND-ORDER ELLIPTIC EQUATIONS

(Presented by Academician M. A. Lavrent' ev, 4 I 1965)

In this note the result obtained in (1) for elliptic equations with two independent variables is generalized to the case of an arbitrary number of dimensions $n > 2$. It turned out that the proof given in (1) can be simplified if one additionally assumes analyticity of the coefficients of the equation. This simpler proof is carried over to the case $n > 2$ with the help of the result of (2) on the local connectedness of real analytic sets.

As was already noted in (1), from the theorem formulated below there follows the uniqueness of the solution of the Cauchy problem with data on an arbitrary set locally dividing the space R^n into two open sets with no common points.

Theorem. Consider the elliptic equation*

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + \sum_{i=1}^{n} a_i(x) u_{x_i} = 0 \tag{1}$$

with coefficients analytic in the domain D. Let $v(x) \in C^1$ in the domain D and be a regular solution of equation (1) in a neighborhood of each point $x \in D$ at which $v(x) \neq 0$. Then $v(x)$ is a regular solution of equation (1) everywhere in the domain D.

The proof rests on the following two lemmas.

Lemma 1. Let $v(x)$ satisfy the conditions of the theorem. Suppose that in a domain $D_1 \subset D$ a regular solution $u(x)$ of the equation $Lu(x) = f(x) > 0$ is given and that the maximum of the function $w(x) = u(x) - v(x)$ is attained at a point $x^0 \in D_1$. Then

$$v(x^0) = 0; \qquad \nabla u(x^0) = 0. \tag{2}$$

Proof of Lemma 1. The regularity of $v(x)$ can be violated only at points where v and ∇v simultaneously vanish. Indeed, if $v(x') = 0$ and $\nabla v(x') \neq 0$, then the set of zeros of the function $v(x)$ in a neighborhood of the point x' coincides with a smooth surface and, consequently, is removable. Thus, if at the point x^0 of maximum of the function $w(x)$ the conditions (2) are not satisfied, then the function $w(x)$ is regular in some neighborhood of the point x^0 and

$Lw(x^0) > 0$. On the other hand, it is known that at the point x^0 of maximum of a function $w(x) \in C^2$ the inequality $Lw(x^0) \le 0$ holds. Lemma 1 is proved.

Lemma 2. Consider a function $u(x)$ analytic in the domain D. Let D_1^0 denote the set of zeros of $\nabla u(x)$ lying in the domain D_1, $\overline{D}_1 \subset D$. Then the image of the set D_1^0 under the mapping $u = u(x)$ consists of a finite number of points.

Lemma 2 is an immediate consequence of Theorem 2 of paper [2] on the local connectedness of the set of zeros of a real analytic function. Below we state this theorem in a somewhat weakened formulation adapted to our purposes.

Lemma 2′. *Let $f(x)$ be an analytic function in a domain D, and let D_0 be the set of zeros of this function. For every point $x^0 \in D_0$ there exists a neighborhood V such that any point $x' \in D_0 \cap V$ can be joined to the point x^0 by a curve*

$$\gamma : \{x = x(t),\ 0 \le t \le 1;\ x(0) = x',\ x(1) = x^0\},$$

lying in $D_0 \cap V$. Moreover, the function $x(t)$ is continuous for $0 \le t \le 1$ and analytic for $0 \le t < 1$.

To prove Lemma 2 it is enough to establish that, for any point $x^0 \in D$, there exists a neighborhood U such that from $\nabla u(x') = 0$, $x' \in U$, it follows that $u(x') = u(x^0)$. The existence of such a neighborhood is obvious if $\nabla u(x^0) \neq 0$. In the case when $\nabla u(x^0) = 0$, it is not difficult to verify that, as U, one may take, in the notation of Lemma 2, the neighborhood V for the function $f(x) = |\nabla u(x)|^2$.

Proof of the theorem. The assertion is of a local character, and it is enough to prove the regularity of $v(x)$ in the ball $\Gamma(\rho)$, $\overline{\Gamma}(\rho) \subset D$, of radius ρ with center at some point of the domain D. The proof will be by contradiction.

Denote by $u^h(x)$ the solution of the Dirichlet problem

$$Lu^h(x) = h = \text{const} \ge 0, \quad x \in \Gamma(\rho); \qquad u^h(x) = v(x), \quad x \in \dot{\Gamma}(\rho).$$

If $v(x) \not\equiv u^0(x)$, then without loss of generality we may assume that the function $w^0(x) = u^0(x) - v(x)$ assumes positive values in the ball $\Gamma(\rho)$. Choose $h > 0$ so small that the function $w^h = u^h - v$ also assumes positive values in the ball $\Gamma(\rho)$. Fix this value of h, and in what follows denote the functions $u^h(x), w^h(x)$ simply by $u(x), w(x)$. The function $w(x)$ is equal to zero on the boundary $\dot{\Gamma}(\rho)$ of the ball $\Gamma(\rho)$ and assumes positive values in the ball $\Gamma(\rho)$. Consequently, the maximum of the function $w(x)$ is attained at some point $x^0 \in \Gamma(\rho)$. Applying Lemma 1, we find that

$$v(x^0) = 0, \quad u(x^0) = w(x^0) > 0.$$

---

* As was indicated in (^{1}), the assumption that the operator Lu contains no term of the form $a(x)u$ is inessential. The general case is reduced (locally) to the case under consideration by replacing $v = \tilde{v}u$, where u is a regular positive solution of the equation $Lu = 0$.

On the other hand, for regular solutions of the equation $Lu > 0$ the maximum principle holds, and therefore there exists a sequence of points x_k such that

$$x_k \to x^0, \quad k \to \infty; \qquad u(x_k) > u(x^0).$$

Now consider the functions

$$w_k(x) = u(x + \xi_k) - v(x); \qquad \xi_k = x_k - x_0.$$

The functions $u_k(x) = u(x + \xi_k)$ satisfy in the ball $\Gamma(\rho - \varepsilon)$, $\varepsilon > 0$ small, for all sufficiently large k, the equation

$$Lu(x + \xi_k) = h + L[u_k(x) - u(x)] = f_k(x)$$

and assume at the point x^0 the value

$$u(x^0 + \xi_k) > u(x^0) > 0.$$

Choose $\varepsilon_0 > 0$ so small and then k_0 so large that for $k > k_0$: a) the point $x^0 \in \Gamma(\rho - \varepsilon_0)$; $|\xi_k| < \varepsilon_0/2$; for $x \in \dot{\Gamma}(\rho - \varepsilon_0)$ the inequality $|u(x + \xi_k) - v(x)| < u(x^0)$ holds; b) the functions $f_k(x)$ are positive in the ball $\Gamma(\rho - \varepsilon_0)$.

It is now easy to verify that, for $k > k_0$, the functions $w_k(x)$, considered in the ball $\Gamma(\rho - \varepsilon_0)$, attain their maximum at an interior point x_k^0, and this maximum exceeds $u(x^0)$. Applying Lemma 1, we find that

$$v(x_k^0) = 0, \quad u(x_k^0 + \xi_k) = w_k(x_k^0) > u(x^0); \quad \nabla u(x_k^0 + \xi_k) = 0. \tag{3}$$

Since the maximum of the function $w(x)$ in the ball $\Gamma(\rho - \varepsilon_0)$ is equal to $u(x^0)$, and the functions $w_k(x) \to w(x)$ as $k \to \infty$, we have

$$u(x_k^0 + \xi_k) \to u(x^0), \quad k \to \infty. \tag{4}$$

From (3) and (4) follows the existence of a sequence of points $x'_k = x_k^0 + \xi_k$, $x'_k \in \Gamma(\rho - \varepsilon_0/2)$, $k > k_0$, with the properties

$$u(x'_k) > u(x^0); \quad \nabla u(x'_k) = 0; \quad u(x'_k) \to u(x^0), \quad k \to \infty.$$

The existence of such a sequence of points for the function $u(x)$, analytic in the ball $\Gamma(\rho)$, contradicts Lemma 2. The theorem is proved.

Institute of Hydrodynamics
Siberian Branch of the Academy of Sciences of the USSR

Received
25 XII 1964

CITED LITERATURE

[1] A. B. Shabat, DAN, **160**, No. 5 (1965).
[2] H. Whitney, F. Bruhat, Comm. Math. Helv., **33**, Fasc. 2 (1959).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.