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Abstract

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ON ONE SIMILARITY PARAMETER

IN THE THEORY OF PLASMA FLOWS

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For considering flows of a fully ionized plasma, the one-fluid approximation of magnetohydrodynamics is, as a rule, insufficient. At the same time, a two-fluid treatment, or its simplified variant—a one-fluid flow with allowance for the Hall effect—makes it possible, at least qualitatively, to describe the principal specific features of plasma dynamics: the frozen-in character of the magnetic field in the electron (and not the ion) fluid, the transport of entropy by electrons, etc. (see, for example, ([1,2])).

The system of equations of the two-fluid model of a fully ionized plasma, in simplified form, is written as ([1])

$$\partial n_i/\partial t + \operatorname{div} n_i \mathbf{v}_i = 0; \qquad (1a)$$

$$\partial n_e/\partial t + \operatorname{div} n_e \mathbf{v}_e = 0; \qquad (1\ )$$

$$\frac{M}{e}\frac{d\mathbf{v}_i}{dt} = -\frac{\nabla p_i}{en_i} + \mathbf{E} + \frac{1}{c}[\mathbf{v}_i, \mathbf{H}] - \frac{\mathbf{j}}{\sigma}; \qquad (1\ )$$

$$\frac{m_e}{e}\frac{d\mathbf{v}_e}{dt} = -\frac{\nabla p_e}{en_e} - \mathbf{E} - \frac{1}{c}[\mathbf{v}_e, \mathbf{H}] + \frac{\mathbf{j}}{\sigma}; \qquad (1\ )$$

$$Mn_iT_i\, dS_i/dt = \chi(T_e - T_i); \qquad (1\ )$$

$$m_en_eT_e\, dS_e/dt = \frac{j^2}{\sigma} - \chi(T_e - T_i); \qquad (1\ )$$

$$\mathbf{j} = e(n_i\mathbf{v}_i - n_e\mathbf{v}_e), \qquad \chi = \frac{3m_en_ek}{M\tau_e}. \qquad (1\ )$$

The notation here is standard.

The manifestation of the difference between the velocities of ions and electrons in Ohm' s law*

$$\frac{\mathbf{j}}{\sigma} = \frac{\nabla p_e}{en} + \mathbf{E} + \frac{1}{c}[\mathbf{v}_e, \mathbf{H}] \tag{2a}$$

or

$$\frac{\mathbf{j}}{\sigma} = \frac{\nabla p_e}{en} + \mathbf{E} + \frac{1}{c}[\mathbf{v}, \mathbf{H}] - \frac{[\mathbf{j}, \mathbf{H}]}{enc} \qquad (2\ )$$

is called the Hall effect. The magnitude of this effect is usually characterized by the Hall parameter

$$\chi = \omega_e \tau_e, \qquad \omega_e = \frac{eH}{m_e c}. \tag{3}$$

In order of magnitude, χ is equal to the ratio of the "Hall term" to the "ohmic" term in equation (2)

$$\chi \sim \frac{|[\mathbf{j}, \mathbf{H}/enc]|}{|\mathbf{j}/\sigma|} \sim \frac{\sigma H}{enc} = \omega_e \tau_e. \tag{4}$$

* Ohm' s law is obtained from (1), (2) by neglecting the mass m_e of the electron, identifying the motion of the ions with the motion of the medium, and setting $n_i = n_e = n$.

However, the quantity χ by itself cannot serve as a measure of the influence of the Hall effect on the flow pattern.

Indeed, if we have a plasma flow for which

$$|[\mathbf{v}, \mathbf{H}]/c| \gg |[\mathbf{j}, \mathbf{H}]/enc|, \tag{5}$$

then even for large χ the plasma dynamics, except perhaps in certain special cases, will not differ substantially from the dynamics of the single-fluid model.

Therefore, when considering sufficiently rapid flows, it is natural, along with χ, to use another dimensionless parameter (1), which in order of magnitude is equal to the ratio of the terms entering into (5):

$$\xi \sim \frac{|[\mathbf{j}, \mathbf{H}]|}{enc\,|[\mathbf{v}, \mathbf{H}]/c|} \sim \frac{|\mathbf{v}_e - \mathbf{v}_i|}{|\mathbf{v}_i|}. \tag{6}$$

This parameter ξ precisely characterizes the relative independence of the ion and electron components of the plasma. In the light of the "standard" set of similarity parameters presently regarded as basic, the parameter ξ is a derived one, equal to the ratio of the Hall parameter to the magnetic Reynolds number:

$$\xi \sim \frac{|[\mathbf{j}, \mathbf{H}]|}{en|[\mathbf{v}, \mathbf{H}]|} = \frac{|[\mathbf{j}, \mathbf{H}]|}{enc \cdot i/\sigma \, |[\mathbf{v}, \mathbf{H}]/c|} \sim \frac{\chi}{R_m}. \tag{7}$$

It is clear, however, that for a well-conducting (in the limit, ideally conducting) plasma, when χ and $R_m \to \infty$, the quantity ξ in fact becomes an independent parameter.

Having made a number of natural assumptions, S. I. Braginskii estimated the quantities $|\mathbf{v}_e - \mathbf{v}_i|$ and v_i and showed that, in order of magnitude (1),

$$\xi \sim 1/\Pi_i^{1/2}, \tag{8}$$

where

$$\Pi_i \equiv 4\pi e^2 nL/Mc^2 \tag{9}$$

is a dimensionless parameter sometimes called the "ion beam parameter." Here L is the characteristic size of the system, and the ions are assumed to be singly charged.

However, both definition (6) and estimate (8) have the drawback that they express the parameter ξ through quantities (v_i, v_e, n) that are often rather difficult to measure and, moreover, have a local character. Therefore, in order to make the quantity ξ a criterion of similarity in the full sense of the word, i.e., to express it through external characteristics, it is necessary to give an integral definition of this quantity.

For the time being, let us restrict ourselves to stationary plasma flows. In that case the continuity equations (1a), (1b) can be satisfied identically if one introduces vector potentials of the ion and electron fluxes,

$$n_e \mathbf{v}_e = \text{rot } \vec{\Psi}_e, \qquad n_i \mathbf{v}_i = \text{rot } \vec{\Psi}_i. \tag{10}$$

Then, using Maxwell' s equation $\text{rot}\,\mathbf{H} = \frac{4\pi}{c}\mathbf{j}$, without loss of generality one may write

$$\mathbf{H} = \frac{4\pi e}{c}\left(\vec{\Psi}_i - \vec{\Psi}_e\right) + \mathbf{H}_0. \tag{11}$$

Here $\mathbf{H}_0$ is the external field in the system, i.e., the field in the absence of plasma.

Fig. 1 and Fig. 2

Figure 1: Fig. 1 and Fig. 2

With the aid of the introduced quantities, the parameter ξ may be defined in the form (cf. (6))

$$\xi = \frac{c\,|\mathbf{H} - \mathbf{H}_0|_{\text{charact}}}{4\pi e\,|\vec{\Psi}_i|_{\text{charact}}}. \tag{12}$$

A similar definition of ξ was used in work (3) in splitting symmetric flows into hydrodynamic and Hall flows. However, for the most interesting case of plasma flows in a channel one can give a more transparent and natural definition of ξ.

Consider, for example, a coaxial channel of arbitrarily varying cross section (Fig. 1). In this case the ion flux per second through the transverse cross section of the channel is

$$\dot{N}_i = \oint_{\Gamma_2} \vec{\Psi}_i\, dl. \tag{13}$$

Here dl is an element of the contour Γ_2. Of course, we assume that

$$\oint_{\Gamma_2} \vec{\Psi}_i\, dl = 0 \tag{13a}$$

and that the inflow of ions occurs over the whole cross section of the channel. If the situation shown in Fig. 2 obtains, then the circulation may be taken along the contour

Fig. 1 **Fig. 2**

I_{max}, for which the flux $\dot{N}_i$ is maximal. For electrons the condition (13a) has no meaning, since the electrons enter not only with the plasma but also along the central metallic conductor*.

If for electrons the situation of Fig. 2 also does not obtain, then

$$\dot{N}_e = \oint_{\Gamma_2} \vec{\Psi}_e\, dl. \tag{14}$$

Using expressions (6), (10), (11), (12), (13), (14), we can give the following final definition of ξ:

$$\xi = \left| \frac{\dot{N}_e - \dot{N}_i}{\dot{N}_i} \right| = \frac{I_p}{I_\Pi}. \tag{15}$$

Fig. 3

Figure 2: Fig. 3

Here $I_p = e(\dot{N}_e - \dot{N}_i)$ is the discharge current in the channel, $\dot{m}$ is the mass flow rate, $I_\Pi = \dot{m}e/M$ is the effective ion current. Definition (15) contains directly measurable quantities I_p and $\dot{m}$; they are easy to determine experimentally. This definition admits natural generalizations to various cases lying outside the framework of the example considered. For example, in considering the acceleration of a "short" plasma bunch in a pulsed injector, the parameter ξ should be understood as the quantity

$$\xi = QM/me, \tag{16}$$

where Q is the charge that has passed through the bunch during the acceleration process; m is the mass of the bunch; M, e are, respectively, the mass and charge of an ion.

In connection with the fact that the vectors $\vec{\Psi}_{i,e}$ generally have three components, it is possible to construct not one but three parameters ξ_α. As an example in which not one but two parameters ξ_z and ξ_φ are of interest, one may cite a coaxial accelerator with a longitudinal field, which for $\xi_\varphi \gg 1$ turns into a Hall accelerator(4) (Fig. 3). Returning to the general consideration of the parameter ξ, we note two classes of phenomena in which the value $\xi \sim 1$ is critical.

* For definiteness, we regard the central electrode as the cathode.

1. **Freezing-in of the field.** As noted above, in the absence of dissipation and as $m_e/M \to 0$, the magnetic field is frozen into the electron component. Therefore, if $\xi \gg 1$ and the flow, for simplicity, is plane, the ratio H/n in the process of plasma flow in a channel may vary arbitrarily (3). Indeed, in this case an intensive replacement takes place of some electrons by others, emitted by the cathode and "carrying" a new value of H/n. Thus, for $\xi \gg 1$, both nondissipative introduction of plasma (more precisely, of the ionic component) into a magnetic field and its withdrawal from a magnetic field are possible. Conversely, for $\xi \ll 1$, the magnetic field is in fact bound to the matter, and a change of H/n without dissipation in the channel is impossible. Since the quantity ξ, simply speaking, shows how many times the electrons compensating the volume charge of the ions have been replaced during the passage of the latter through the channel, it is natural to call the quantity ξ the parameter or number of exchange.

Fig. 3

2. **Transfer of entropy by electrons.** Electron entropy behaves in an analogous manner. If $\xi \gg 1$, then the entropy of the electrons entering

the plasma from the cathode is transferred to the anode and therefore may be distributed quite arbitrarily along the length of the channel. If, however, $\xi \ll 1$ and dissipation is absent, the entropy of the electrons along the channel is conserved.

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Note: Figure translations are in progress. See original paper for figures.