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Abstract

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SOME PROPERTIES OF FUNCTIONS REGULAR IN A MULTIPLY CONNECTED DOMAIN

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The note proposes a method for studying extremal properties of functions regular in a multiply connected domain, which goes back to the well-known symmetrization principle of W. K. Hayman ([1]) and is a generalization of the symmetrization principle for an annulus ([2]).

1. Let G be a domain of the open z-plane whose boundary includes n components—analytic, pairwise disjoint, closed Jordan curves $\gamma_1, \gamma_2, \dots, \gamma_n$ (these are not all the boundary components of the domain G), and suppose that the finite simply connected domains bounded by these curves have no common points. We shall also assume that the set of boundary points of the domain G not belonging to the curves $\gamma_1, \gamma_2, \dots, \gamma_n$ contains no limit points belonging to $\gamma_1, \gamma_2, \dots, \gamma_n$. In what follows, we shall denote the boundary of the domain everywhere by ∂G, the aggregate of the curves $\gamma_1, \gamma_2, \dots, \gamma_n$ by γ, and $\partial G - \gamma$ by β. Let E_1 be the part of the complement of the domain G in the plane bounded by γ, and let E_0 be the remainder of this complement.

Definition 1. A **potential function for the domain G with respect to the system** $\{E_0, E_1\}$ will mean the least nonnegative function harmonic in the domain G, continuous in the whole plane, and equal to one on E_1.

With the aid of an exhausting sequence of domains, and relying on Garnak’ s theorem, it is not difficult to prove the existence and uniqueness of the potential function for an arbitrary domain G with respect to the system $\{E_0, E_1\}$.

Definition 2. A domain G of the open plane is called **admissible with respect to the system** $\{E_0, E_1\}$ if for it there exists a potential function equal to zero on E_0.

Definition 3. The **capacity of the domain G with respect to the system** $\{E_0, E_1\}$ is the quantity

$$C\{G, E_0, E_1\} = \int_\gamma \left| \frac{\partial \omega}{\partial h} \right| ds, \tag{1}$$

where $\partial \omega / \partial h$ is the derivative, in the direction of the normal to γ directed into the domain G, of the potential function $\omega(z)$, and ds is the element of arc. In the case of an admissible domain G, the notion of capacity thus defined coincides with the capacity of a condenser ([1,3]).

Denote by $\mathfrak{M}(G)$ the aggregate of single-valued functions $w = f(z)$, regular in the domain G, satisfying the following conditions: 1) the function $w = f(z)$ maps the curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ onto pairwise nonintersecting closed analytic Jordan curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$, and the finite domains bounded by $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ have no common points with $G_f = f(G)$; 2) under a single traversal of the curves $\gamma_1, \gamma_2, \ldots, \gamma_n$ in the po-

in the positive direction relative to the domain G, the curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ are traversed a finite number of times.

Denote by $p_1, \ldots, p_m$ the number of circuits in the positive direction, relative to the image of the domain G, of the curves $\Gamma_1, \ldots, \Gamma_m$. Let E_1^f be the part of the complement of the domain G_f in the plane bounded by the curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$, and let E_0^f be the remaining part of the complement.

Theorem 1. If the function $w = f(z) \in \mathfrak{M}(G)$, then

$$pC\{G_f, E_0^f, E_1^f\} \le C\{G, E_0, E_1\}, \tag{2}$$

where $p = \min\{p_1, p_2, \ldots, p_m\}$. If it is additionally known that the domain G is admissible relative to $\{E_0, E_1\}$, then equality in (2) can occur only in the case when each point $w \in G_f$ is the image of exactly p points of G, and the domain G_f is admissible relative to $\{E_0^f, E_1^f\}$.

Special cases of Theorem 1 are M. Schiffer' s lemma ([4]), a theorem established by the author ([2]), and also a result of M. Schiffer proved by means of D. Jenkins' s extremal length method ([5]).

2. Let G be an arbitrary domain of the open z-plane whose complement in the plane consists of a closed bounded set E_1 and a closed unbounded set E_0 not intersecting E_1.

Definition 4. We shall call the domain G **weakly admissible with respect to the system** $\{E_0, E_1\}$ if there exists in the domain G a harmonic function, continuous in the entire z-plane, equal to one on E_1 and to zero on E_0.

Denote by $\mathfrak{A}(G)$ the totality of transformations of the system $\{G, E_0, E_1\}$ into a system $\{G^*, E_0^*, E_1^*\}$ satisfying the following conditions: 1) under a transformation $A \in \mathfrak{A}(G)$, every domain $G' \subset G$, bounded by closed analytic curves and admissible relative to the system $\{E_0, E_1\}$, is transformed into a domain $G'^* \subset G^*$ weakly admissible relative to the transformed system $\{E_0'^*, E_1'^*\}$; 2) the capacities of the domain $G' \subset G$, admissible relative to the system $\{E_0', E_1\}$, bounded by closed analytic curves, and of the transformed domain G'^* satisfy the inequality

$$C\{G'^{*}, E_0'^{*}, E_1^{*}\} \leq C\{G', E_0', E_1\};$$

3) if the domains G_1 and G_2 satisfy the condition $G_1 \subset G_2 \subset G$, and G_i is admissible with respect to the system $\{E_0^{(i)}, E_1\}$, $i = 1, 2$, then $G_1^* \subset G_2^* \subset G$.

Theorem 2. If the transformation $A \in \mathfrak{A}(G)$, then

$$C\{G^*, E_0^*, E_1^*\} \leq C\{G, E_0, E_1\}. \tag{3}$$

Combining the results established in Theorems 1 and 2, we obtain the following

Principle of A-transformation for the class $\mathfrak{M}(G)$. If the function $w = f(z) \in \mathfrak{M}(G)$, then the inequality

$$pC\{G_f^*, E_0^{f*}, E_1^{f*}\} \leq C\{G, E_0, E_1\}. \tag{4}$$

is valid.

Suppose, in addition, that the domain G is known to be admissible relative to $\{E_0, E_1\}$. Then equality in (4) can occur only in the case when each point of the domain G_f is the image of exactly p points of G under the mapping $w = f(z)$. In this case the domain G_f is admissible relative to the system $\{E_0^f, E_1^f\}$.

3. With the aid of Theorem 1, and also of the principle of A-transformation just formulated, one can establish a number of extremal properties of functions of the class $\mathfrak{M}(G)$ and of its various subclasses. We give, as an example, several theorems.

For convenience we introduce a number $R > 0$ such that $C\{G, E_0, E_1\} = 2\pi/\ln R$. Denote by $\mathfrak{R}(G)$ the class of functions $w = f(z) \in \mathfrak{M}(G)$ that take the curves $\gamma_1, \gamma_2, \dots, \gamma_n$ onto the circle $|w| = 1$, traversed p times under a single traversal of γ in the positive direction.

Theorem 3. If $w = f(z) \in \mathfrak{R}(G)$, then the area S of the domain G_f satisfies the inequality

$$S \geq \pi(R^{2p} - 1). \tag{5}$$

If the domain G is admissible, then equality in (5) can be attained only in the case of a function $w = f(z)$ mapping G onto the p-fold covered annulus $1 < |w| < R^p$.

In what follows we shall assume that the origin is an interior point of the set E_1^f.

Theorem 4. If $w = f(z) \in \mathfrak{R}(G)$, then

$$\sup|f(z)|/\inf|f(z)| \geq R^p, \quad z \in G.$$

Let D_A be a doubly connected domain lying in the z-plane and satisfying the following conditions: 1) the origin and the point at infinity belong to different components of the complement of the domain to the plane (the origin is considered only for definiteness); 2) the Riemann modulus $m(D_A)$ of the domain D_A (see, for example, ([6])) is not less than R; 3) the result of some transformation $A \in \mathfrak{A}(D_A)$ of the domain D_A coincides with D_A.

Denote the inner boundary of the domain D_A by K, and the outer boundary by L.

Let A be symmetrization of the domain with respect to the positive real axis ([1]), and let $D_0(R)$ be a doubly connected domain whose Riemann modulus is equal to R, obtained from D_A by removing some rectilinear segment lying on the negative part of the real axis, one endpoint of which belongs to L. The distance from the other endpoint of this segment to the origin will be denoted by $\rho(D_A, R)$.

Theorem 5. If $w = f(z) \in \mathfrak{R}(G)$, $m = 1$, and the domain G_f^*—the result of symmetrization of G_f with respect to the positive real axis—lies in D_A, with $m(D_A) \geq R^p$, A is symmetrization with respect to the positive real axis, then the upper bound R_f of the radii of circles with center at the origin, lying entirely in the domain $G_f \cup E_1^f$, satisfies the inequality

$$R_f \geq \rho(D_A, R^p). \tag{6}$$

If the domain G is admissible, then equality in (6) can be attained only in the case of a function $w = f(z) \in \mathfrak{R}(G)$ mapping G onto the p-fold covered domain G_f, and if the domain G_f^* is weakly admissible, then G_f, up to rotation about $w = 0$, must coincide with $D_0(R^p)$.

Theorem 5 contains as special cases the corresponding results of Gretsch ([7]), T. Kubo ([8]), and also of the author ([2]).

It is not hard to see that the collection of transformations of the class $\mathfrak{A}(G)$ includes completion transformations, consisting in adjoining to the domain G certain connected components of the complement of G to the plane.

Theorem 6. If $w = f(z) \in \mathfrak{R}(G)$, $m = 1$, and G_f^* is a doubly connected domain bounded by E_1^f and the unbounded connected component of the set E_0, then

$$m(G_f^*) \geq R^p. \tag{7}$$

If the domain G is admissible, then equality in (7) can occur only in the case when $G_f^* = G_f$, and every point of the domain G_f is the image of exactly p points of G under the mapping $w = f(z)$.

Theorem 6 is an immediate generalization of a well-known result of M. Schiffer [4] and makes it possible to extend to the class $\mathfrak{M}(G)$ and its various subclasses many covering theorems known for functions univalent in a ring. However, in doing so, just as when using Schiffer's lemma, the internal boundary components of the domain are not taken into account.

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Note: Figure translations are in progress. See original paper for figures.