ON THE DEPENDENCE ON THE BOUNDARY OF SOLUTIONS OF SOME MIXED PROBLEMS FOR THE EQUATIONS OF SMALL OSCILLATIONS OF A ROTATING FLUID

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Abstract

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ON THE DEPENDENCE ON THE BOUNDARY OF SOLUTIONS OF SOME MIXED PROBLEMS FOR THE EQUATIONS OF SMALL OSCILLATIONS OF A ROTATING FLUID

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Consider the system of equations describing small oscillations of a rotating fluid:

$$\frac{\partial u}{\partial t} = v - \frac{\partial p}{\partial x}; \quad \frac{\partial v}{\partial t} = -u - \frac{\partial p}{\partial y}; \quad \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z}; \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1}$$

S. L. Sobolev proved the existence and uniqueness of the solution of (1) satisfying the initial conditions

$$u\big|_{t=0} = u_0(x,y,z); \qquad v\big|_{t=0} = v_0(x,y,z); \qquad w\big|_{t=0} = w_0(x,y,z) \tag{2}$$

$$\left(\mathbf{U}\big|_{t=0} = \mathbf{U}_0(x,y,z)\right)$$

and one of the boundary conditions:

$$p\big|_{\Gamma} = 0; \tag{3}$$

$$u\cos(n,x) + v\cos(n,y) + w\cos(n,z)\big|_{\Gamma} = 0, \tag{4}$$

where Γ is the boundary of the domain Ω in the space x, y, z. In the case of the boundary condition (4), the vector $\mathbf{U} = (u, v, w)$ is determined uniquely, and p up to an addend depending only on t. The behavior of the solutions of problem (1), (2), (3) in the plane case (when Ω is an infinite cylinder with generator parallel to the y-axis; $\partial p/\partial y \equiv 0$) was investigated in ([3,4,7]). In this case, in ([3]), examples were constructed of domains arbitrarily close to a circle in which there exist solutions that are not almost periodic, although in the case of the circle all solutions are almost periodic. If $\mathbf{U}, p$ satisfy (1)–(3), then p is a solution of the problem

$$\frac{\partial^2}{\partial t^2}\Delta p + \frac{\partial^2 p}{\partial z^2} = 0; \tag{5}$$

$$p\Big|_{t=0} = p_0(x,y,z); \qquad \frac{\partial p}{\partial t}\bigg|_{t=0} = p_1(x,y,z); \tag{6}$$

$$p\Big|_{\Gamma} = 0, \tag{7}$$

where p_0, p_1 are determined uniquely by $\mathbf{U}_0$. In the case of condition (4), (7) is replaced by the following boundary condition:

$$\frac{\partial^2}{\partial t^2}\frac{dp}{dn} + \frac{\partial}{\partial t}\left[\frac{\partial p}{\partial y}\cos(n,x) - \frac{\partial p}{\partial x}\cos(n,y)\right] + \frac{\partial p}{\partial z}\cos(n,z)\bigg|_{\Gamma} = 0. \tag{8}$$

In the plane case, condition (8) has the form

$$\frac{\partial^2}{\partial t^2}\frac{dp}{dn} + \frac{\partial p}{\partial z}\cos(n,z)\bigg|_{\gamma} = 0, \tag{9}$$

where γ is the boundary of the projection D of the cylinder Ω onto the plane $y = 0$. In ([7,8]) an infinite-dimensional subspace of the space $\dot{W}_2^1(D)$ was constructed, a basis in which is formed by functions of the form

$$f(x + \mu_{k,1}^N(x,z)z) + g(x + \mu_{k,2}^N(x,z)z) \quad (k = 0, \dots, N-1;\ N = 2, 3, \dots),$$

and the solutions of the mixed problem (1)–(3) in the planar case, for initial data from this subspace, are linear combinations of functions of the form

$$\int_{\mu_{k,1}^N(x,z)}^{\mu_{k,2}^N(x,z)} f(\alpha)e^{i\lambda(\alpha)t}\,d\alpha \tag{10}$$

and of almost periodic functions of t.

The functions $\mu_{k,i}^N$ depend continuously on the domain. S. L. Sobolev [2] established continuous dependence on the domain, on a finite time interval, of the solutions of problem (1)–(3), by estimating the norm of the difference between the solution (U, p) in the domain Ω and the solution constructed in the domain Ω_1, containing Ω, which satisfies the initial condition

$$U_1\big|_{t=0} = \begin{cases} U\big|_{t=0}, & \text{if } (x, y, z) \in \Omega, \\ 0, & \text{if } (x, y, z) \in \Omega_1 - \Omega. \end{cases}$$

It follows from this estimate that the indicated difference in Ω is small, provided only that the boundaries of the domains Ω and Ω_1 are sufficiently close; moreover, the order of convergence of this difference to zero as the boundaries approach one another depends on U_0.

Consider the planar case of problem (1)–(3). Let D_ε be a one-parameter family of simply connected domains in the x, z-plane with boundaries γ_ε. For simplicity, consider the case where γ_0 is a circle, $\gamma_\varepsilon \to \gamma_0$. Suppose γ_ε is given by the equations $x = x_\varepsilon(t)$, $y = y_\varepsilon(t)$. Require that

$$\|x_0 - x_\varepsilon\|_{C_{k+\alpha}} \xrightarrow[\varepsilon\to 0]{} 0, \quad \|y_0 - y_\varepsilon\|_{C_{k+\alpha}} \xrightarrow[\varepsilon\to 0]{} 0,$$

where k is some natural number. Map the domains D_ε conformally onto D_0 by means of the function

$$f_\varepsilon(\zeta) = \zeta \exp\{u_\varepsilon + iv_\varepsilon\},$$

where $\zeta = x + iz$, $\Delta u_\varepsilon = 0$,

$$u_\varepsilon\big|_{\gamma_\varepsilon} = -\ln\sqrt{x^2 + y^2}\big|_{\gamma_\varepsilon}, \quad v_\varepsilon = \int_{(0,0)}^{(x,z)} \left(-\frac{\partial u_\varepsilon}{\partial x}\,dz + \frac{\partial u_\varepsilon}{\partial z}\,d\xi\right).$$

By virtue of our assumptions, using the estimates from [6], we obtain

$$\|u_\varepsilon\|_{C_{k+\alpha}(D_\varepsilon)} \le K\delta(\varepsilon); \quad \|v_\varepsilon\|_{C_{k+\alpha}(D_\varepsilon)} \le K\delta(\varepsilon). \tag{11}$$

In (11), K can be chosen independent of ε, if

$$\|x_\varepsilon\|_{C_{k+1}} \le \widetilde{K}, \quad \|y_\varepsilon\|_{C_{k+1}} \le \widetilde{K},$$

where $\widetilde{K}$ does not depend on ε.

In the coordinates

$$\xi = \operatorname{Re} f_\varepsilon(\zeta) = \varphi_\varepsilon(x, z); \quad \eta = \operatorname{Im} f_\varepsilon(\zeta) = \psi_\varepsilon(x, z) \tag{12}$$

equation (5) has the form

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 p}{\partial \xi^2} + \frac{\partial^2 p}{\partial \eta^2}\right) + \frac{1}{I_\varepsilon} L_1^\varepsilon p = 0, \tag{13}$$

where

$$I_\varepsilon = (\partial\varphi_\varepsilon/\partial x)^2 + (\partial\varphi_\varepsilon/\partial z)^2, \tag{14}$$

and $L_1^\varepsilon p$ is the expression for $\partial^2 p/\partial z^2$ in the new coordinates.

Introduce the operator Δ^{-1}: $\Delta^{-1}(\partial^2 p/\partial\xi^2 + \partial^2 p/\partial\eta^2) = p$, if $p|_{\gamma_0} = 0$. Then (13), under condition (7), can be written in the form

$$\frac{\partial^2 p}{\partial t^2} = -\Delta^{-1}\left(\frac{1}{I_\varepsilon} L_1^\varepsilon p\right) = A_\varepsilon p. \tag{15}$$

For any ε, the operator A_ε is a self-adjoint bounded operator in the space $\overset{\circ}{W}{}_2^2(D_0)$ with scalar product

$$[u, v] = \iint_{D_0} \left(\frac{\partial u}{\partial\xi}\frac{\partial v}{\partial\xi} + \frac{\partial u}{\partial\eta}\frac{\partial v}{\partial\eta}\right) d\xi\, d\eta.$$

It is also obvious that $-[p,p] \le [A_\varepsilon p, p] \le 0$.

Theorem 1. Suppose

$$\|x_0(t) - x_\varepsilon(t)\|_{C_{1+\alpha}} \xrightarrow[\varepsilon\to 0]{} 0; \quad \|x_\varepsilon\|_{C_2} \le K_1; \quad \|y_0(t) - y_\varepsilon(t)\|_{C_{1+\alpha}} \xrightarrow[\varepsilon\to 0]{} 0; \quad \|y_\varepsilon\|_{C_2} \le K_1.$$

Then

$$\|A_\varepsilon - A_0\| \xrightarrow[\varepsilon\to 0]{} 0,$$

if p_ε and p_0 are solutions of equation (15), respectively for arbitrary ε and $\varepsilon = 0$, constructed from the same initial data; then for any finite T, $-T \le t \le T$, the estimate holds

$$\|p_\varepsilon - p_0\|_{\overset{\circ}{W}{}_2^2(D_0)} \le C(\varepsilon, T)\left\{\left\|p_0\big|_{t=0}\right\|_{\overset{\circ}{W}{}_2^2(D_0)} + \left\|\frac{\partial p}{\partial t}\bigg|_{t=0}\right\|_{\overset{\circ}{W}{}_2^2(D_0)}\right\},$$

where $C(\varepsilon, T) \xrightarrow[\varepsilon\to 0]{} 0$.

The proof is obvious if one uses the equality

$$[(A_\varepsilon - A_0)p, p] = \iint_{D_0}\left[\left(\frac{\partial p}{\partial\eta}\right)^2 - \left(\frac{\partial p}{\partial\xi}\frac{\partial\varphi_\varepsilon}{\partial z} + \frac{\partial p}{\partial\eta}\frac{\partial\psi_\varepsilon}{\partial z}\right)^2 I_\varepsilon^{-1}\right] d\xi\, d\eta$$

and represents the solutions in the form of a power series in t.

Theorem 2. Suppose

$$\|x_\varepsilon\|_{C_3} \le K_2; \qquad \|y_\varepsilon\|_{C_3} \le K_2;$$

$$\|x_\varepsilon(t) - x_0(t)\|_{C_{2+\alpha}} \xrightarrow[\varepsilon\to 0]{} 0; \qquad \|y_\varepsilon(t) - y_0(t)\|_{C_{2+\alpha}} \xrightarrow[\varepsilon\to 0]{} 0.$$

Suppose $p_0(t)$ is a solution of equation (13) for $\varepsilon = 0$, satisfying the boundary condition (9). Then there exists a solution $p_\varepsilon(t)$ of equation (13), satisfying the conditions

$$p_\varepsilon(0) = p_0(0); \qquad \left.\frac{\partial p_\varepsilon}{\partial t}\right|_{t=0} = \left.\frac{\partial p_0}{\partial t}\right|_{t=0}$$

and the boundary condition obtained from (9) by replacing (12) by

$$\frac{\partial^2}{\partial t^2} M_1^\varepsilon p_\varepsilon + M_2^\varepsilon p_\varepsilon\Big|_{\gamma_0} = 0 \tag{16}$$

and such that

$$\|p_\varepsilon(t) - p_0(t)\|_{C_{2+\alpha}(D_0)} \le C_1(\varepsilon, T)\left\{\|p_0(0)\|_{C_{2+\alpha}(D_0)} + \left\|\left.\frac{\partial p_0}{\partial t}\right|_{t=0}\right\|_{C_{2+\alpha}(D_0)}\right\}$$

for $-T \le t \le T$, where $C_1(\varepsilon, T) \xrightarrow[\varepsilon\to 0]{} 0$.

A solution of equation (13) satisfying the conditions specified in the theorem can be represented in the form of the series

$$p_\varepsilon = \sum_0^\infty p_i^\varepsilon(\xi, \eta) t^i,$$

where the functions p_i^ε for $i > 2$ are orthogonal functions, identically equal to one. It is not difficult to see that in this case p_i^ε are determined uniquely from the equation and boundary conditions, and for them an estimate is valid which guarantees uniform convergence of this series in $C_{2+\alpha}(D_0)$ on every finite time interval. Thus, to prove the theorem it suffices to compare the partial sum of the series for $p_0(t)$ with the corresponding partial sum for $p_\varepsilon(t)$, and the inequality appearing in the statement of Theorem 2 is obtained after applying the a priori estimates from (6).

Let now Ω_ε be a one-parameter family of domains in the space x, y, z, star-shaped with respect to one and the same point x_0, y_0, z_0. Let Γ_ε be given by the equations $\zeta = \zeta_\varepsilon(\varphi, \theta)$, where (φ, θ, ζ) are the spherical coordinates of the point x, y, z. If $\|\zeta_\varepsilon\|_{C_3} \le K_6$, K_6 is a constant independent of ε, and $\|\zeta_\varepsilon - \zeta_0\|_{C_{2+\alpha}} \to 0$ $(\varepsilon \to 0)$, then Ω_ε is mapped onto Ω_0 by the mapping

$$\xi = \xi_\varepsilon(x, y, z), \qquad \eta = \eta_\varepsilon(x, y, z), \qquad \zeta = \zeta_\varepsilon(x, y, z),$$

where

$$\|\xi_\varepsilon - x\|_{C_{2+\alpha}}, \quad \|\eta_\varepsilon - y\|_{C_{2+\alpha}}, \quad \|\zeta_\varepsilon - z\|_{C_{2+\alpha}}$$

are small together with ε. The equation

(1) is then written in the form

$$\frac{\partial^2}{\partial t^2} N_\varepsilon p_\varepsilon + E_\varepsilon p_\varepsilon = 0,$$

and condition (4) in the form

$$\frac{\partial^2}{\partial t^2} \Phi_1^\varepsilon p_\varepsilon + \frac{\partial}{\partial t} \Phi_2^\varepsilon p_\varepsilon + \Phi_3^\varepsilon p_\varepsilon \big|_{\Gamma_0} = 0.$$

Analogously to the two-dimensional case, it is proved that, under the restrictions imposed,

$$\|p_\varepsilon(\xi, \eta, \zeta, t) - p_0(\xi, \eta, \zeta, t)\|_{C_{2+\alpha}(\Omega_0)} \leq$$

$$\leq K(T, \varepsilon) \left\{ \|p_0|_{t=0}\|_{C_{2+\alpha}(\Omega_0)} + \left\| \frac{\partial p_0}{\partial t}\bigg|_{t=0} \right\|_{C_{2+\alpha}(\Omega_0)} \right\},$$

where $p_\varepsilon|_{t=0} = p_0|_{t=0}$, $\partial p_\varepsilon/\partial t|_{t=0} = \partial p_0/\partial t|_{t=0}$; $-T \leq t \leq T$, and p_ε is a solution of problem (1), (2), (3), or the correspondingly chosen solution of problem (1), (2), (4).

Thus it has been established that (U, p), the solution of problem (1)–(3), changes little on each finite time interval under small changes of the boundary. In the case of problem (1), (2), (4), the vector U changes little. Since the solution depends continuously on the initial data [(5)], the problem is well posed in the classical sense. However, as S. L. Sobolev observed, the existence of solutions (10) in the case $\partial \mu_{k,l}^N / \partial z \neq 0$ shows that the problem is not suitable for describing the physical process for large t [(2)]. Indeed, in this case U is expressed through $\operatorname{grad} p$, and for sufficiently large t the difference $\|U(x_0, z_0, t) - U(x, z, t)\|$ is not an infinitely small quantity as (x, z, t) tends to (x_0, z_0, ∞).

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Note: Figure translations are in progress. See original paper for figures.