# THE EXISTENCE IN THREE-VALUED LOGIC OF A CLOSED CLASS WITH A FINITE BASIS THAT HAS NO FINITE COMPLETE SYSTEM OF IDENTITIES

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**Abstract**

**Full Text**

**MATHEMATICS**

**V. L. MURSKII**

# THE EXISTENCE IN THREE-VALUED LOGIC OF A CLOSED CLASS WITH A FINITE BASIS THAT HAS NO FINITE COMPLETE SYSTEM OF IDENTITIES

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**1.** As R. C. Lyndon showed ([1]), every algebra with 2 elements has a finite complete system of identities. The same author constructed ([2]) an example of an algebra with 7 elements (in other words, a closed class of functions of 7-valued logic) that has no finite complete system of identities. V. V. Vysnin constructed ([3]) an analogous example for 4-valued logic.

In the present note an example is constructed of an algebra with 3 elements, generated by a single binary function and having no finite complete system of identities.

Definitions of the notions of algebra, formula, subformula, identity, and completeness of a system of identities may be found, for example, in ([1,4]). Let $A$ be the algebra with elements $0, 1, 2$ and one binary operation, denoted by $xy$ and defined by Table 1.

**Table 1**

| $x\backslash y$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 |
| 2 | 0 | 2 | 2 |

**Theorem.** *In the algebra $A$ there exists no finite complete system of identities.*

In what follows, when speaking of formulas, functions, identities, etc., we shall always mean formulas, functions, and identities of the algebra $A$.

If $\varphi_1, \varphi_2, \ldots, \varphi_k$ are arbitrary formulas, and $F$ is a formula all of whose variables are among $y_1, y_2, \ldots, y_k$, then, as usual, we shall denote by

$$S^{y_1 y_2 \cdots y_k}_{\varphi_1 \varphi_2 \cdots \varphi_k} F$$

the result of substituting the formulas $\varphi_1, \varphi_2, \dots, \varphi_k$ into the formula $F$ in place of the variables $y_1, y_2, \dots, y_k$, respectively. Let $f = g$ be an identity containing no variables other than $y_1, y_2, \dots, y_k$, and let $\Phi$ be a formula containing a subformula $\Phi' \subset S^{y_1 \cdots y_k}_{\varphi_1 \cdots \varphi_k} f$ (or $\Phi' \subset S^{y_1 \cdots y_k}_{\varphi_1 \cdots \varphi_k} g$). Let $\Psi$ be the formula obtained from $\Phi$ by replacing the subformula* $\Phi'$ by $S^{y_1 \cdots y_k}_{\varphi_1 \cdots \varphi_k} g$ (respectively by $S^{y_1 \cdots y_k}_{\varphi_1 \cdots \varphi_k} f$). We shall say that $\Psi$ is obtained from $\Phi$ by application of the identity $f = g$. To prove the theorem it is enough to indicate, for every finite system of identities, two equivalent formulas (i.e., expressing equal functions) which cannot be obtained one from the other by applications of the identities of the given system. We shall prove that for any $n \geq 3$ the formulas

$$F_n \subset x_1(x_2(x_3 \dots (x_{n-1}(x_n x_1)) \dots)),$$

$$G_n \subset (x_1 x_2)(x_n(x_{n-1} \dots (x_4(x_3 x_2)) \dots))$$

are equivalent, but are not transformed one into the other by applications of identities in which every formula contains occurrences of no more than $n-1$ distinct variables.

**2. Adjacent occurrences. Equivalent formulas.** In what follows the term *occurrence* will mean an occurrence of a variable in a formula. Two occurrences $b_1, b_2$ of a formula $F$ are called adjacent (in $F$),

* When speaking of a subformula, we shall always mean a fixed occurrence of the subformula in the formula.

if $F$ contains a subformula of the form $F_1 F_2$ and $b_1$ is the leftmost $*$ occurrence in $F_1$, while $b_2$ is in $F_2$, or conversely. Two variables $x$ and $y$ (possibly coinciding) are called adjacent in the formula $F$ if in $F$ there is an occurrence of $x$ adjacent to an occurrence of $y$.

**Lemma 1.** *Every formula $F$ of the algebra $A$ expresses a function essentially depending on all variables occurring in $F$.*

**Proof.** Let $x_1, x_2, \dots, x_s$ be all the variables in $F$. It follows from Table 1 that if at least one of these variables has value 0, then the value of the formula $F$ is 0, and that if the values of all variables are equal to 2, then the value of the formula $F$ is equal to 2. Thus, for any $i$, $F$ has value 0 when

$$x_1 = \dots = x_{i-1} = x_{i+1} = \dots = x_s = 2, \quad x_i = 0$$

and value 2 when $x_1 = \dots = x_s = 2$, which proves the lemma.

**Corollary.** *Equivalent formulas have the same set of variables.*

**Lemma 2.** a) *If the value of a formula is different from* 0 *(for given values of the variables), then it is equal to the value of the left variable of this formula;* b) *the value of a formula $F$ is equal to* 0 *if and only if either some variable in $F$ is equal to* 0*, or in $F$ there is a pair of adjacent variables equal to* 1 *(possibly coinciding).*

**Proof.** a) follows from the fact that for any $a, b$, from $ab \neq 0$ it follows that $ab = a$.

b) **Sufficiency.** If in $F$ some variable is equal to 0, then $F = 0$, since the result of "multiplication" by 0 is always 0; if, however, two adjacent variables in $F$ are equal to 1, then in $F$ there is a subformula $F_1F_2$ such that the left variables of the subformulas $F_1$ and $F_2$ are equal to 1. If, for the values of the variables under consideration, $F_1 = 0$ or $F_2 = 0$, then everything is proved; otherwise, by the already proved part a) of Lemma 2, $F_1 = 1$ and $F_2 = 1$, whence $F_1F_2 = 0$ and $F = 0$.

**Necessity.** Let $F = 0$, and let $F'$ be a subformula of the formula $F$ whose value is equal to 0, but the values of all proper (i.e. different from $F'$) subformulas are not equal to 0. If $F'$ is a variable, then it is equal to 0, and what is required is proved; otherwise $F' = F_1F_2$, $F_1F_2 = 0$, $F_1 \neq 0$, $F_2 \neq 0$. Hence $F_1 = F_2 = 1$ and, by Lemma 2a), the left variables of the subformulas $F_1$ and $F_2$ are equal to 1. But they are adjacent in $F$. Lemma 2 is proved.

It follows from the lemma that the function expressed by a formula $F$ is completely determined if the left variable of the formula $F$ and all pairs of adjacent variables in $F$ are specified. In other words, if two formulas have the same left variables and all pairs of adjacent variables, then these formulas are equivalent. The following lemma is a converse of the last assertion under certain restrictions.

**Lemma 3.** *Let $F$ be a formula in which no variable is adjacent to itself. Then every formula $G$ equivalent to $F$ has the same left variable and the same set of pairs of adjacent variables as $F$.*

**Proof.** Let $z_1, \dots, z_s$ be the variables of $F$ (by the corollary to Lemma 1, $G$ contains exactly the same variables). First of all, in $G$ no variable $x_i$ is adjacent to itself: otherwise, when $z_i = 1$ and

$$z_1 = \dots = z_{i-1} = z_{i+1} = \dots = z_s = 2,$$

we would have, by Lemma 2, $F \neq 0$, $G = 0$. Next, if the left variable $z_j$ of the formula $F$ did not coincide with the left variable of the formula $G$, then when $z_j = 1$ and

$$z_1 = \dots = z_{j-1} = z_{j+1} = \dots = z_s = 2$$

we would have $F = 1$, $G = 2$ (by Lemma 2 and the absence in $F$ and $G$ of variables adjacent to themselves). Finally, if two variables $z_i, z_j$, $i > j$, adjacent in $F$, were not adjacent in $G$, then when

$$z_i = z_j = 1, \quad z_1 = \dots = z_{i-1} = z_{i+1} = \dots = z_{j-1} = z_{j+1} = \dots = z_s = 2$$

we would have $F = 0$, $G \neq 0$, and analogously from $G$ to $F$. The lemma is proved.

**3. Formulation of the main lemma. Some auxiliary assertions.** Let $K_n$ be the class of all formulas from

* In what follows, the leftmost occurrence will be called simply the left one.

variables $x_1, x_2, \dots, x_n$, in which the left variable is $x_1$, and the pairs of adjacent variables are all the pairs $x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_1x_n$, and only these pairs. By Lemmas 2 and 3, all formulas from $K_n$ are equivalent, and every formula equivalent to a formula from $K_n$ itself belongs to $K_n$. The class $K_n$ contains, in particular, the formulas $F_n$ and $G_n$ defined in Sec. 1.

Let us define the following property $P_n$ of formulas of the algebra $A$: a formula $F$ has property $P_n$ if it belongs to the class $K_n$ and in $F$ there is an occurrence of the variable $x_2$ adjacent simultaneously to some occurrence of the variable $x_1$ and to some occurrence of the variable $x_3$.

**Lemma 4 (main).** *Let $\Phi, \varphi_1, \dots, \varphi_k$ be formulas; $f, g$ equivalent formulas in the variables $y_1, \dots, y_k$; $\Phi' = S^{y_1 \dots y_k}_{\varphi_1 \dots \varphi_k} f$, $\Psi' = S^{y_1 \dots y_k}_{\varphi_1 \dots \varphi_k} g$. Let, further, $\Phi$ contain $\Phi'$ as a subformula, and let $\Psi$ be the formula obtained from $\Phi$ by replacing the subformula $\Phi'$ by $\Psi'$. Finally, let $k \le n-1$, and let $\Phi$ have property $P_n$. Then $\Psi$ also has property $P_n$.*

From the main lemma, in view of what was said at the end of Sec. 1, the theorem being proved follows: it is verified directly that $F_n$ has property $P_n$, whereas $G_n$ does not (although $G_n \in K_n$).

For the proof of the main lemma we shall need auxiliary Lemmas 5-7, whose proofs are not difficult and are omitted here.

**Lemma 5.** *Let $P$ be a word in the alphabet $\{x_1, x_2, \dots, x_n\}$, beginning with $x_1$, ending with $x_3$, and not containing some letter $x_i$, $i \ne 2$. Let, further, every (unordered) pair of neighboring letters of the word $P$ be one of the pairs $x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_1x_n$. Then $P$ contains the subword $x_1x_2x_3$.*

**Lemma 6.** *Let $F$ be a formula, $F'$ its subformula, $b_1$ an occurrence in $F'$ which is not left for $F'$, and $b_2$ an occurrence adjacent to $b_1$. Then $b_2$ lies in $F'$.*

**Corollary 1.** *Let $b_2$ be an occurrence in a formula $F$, adjacent to the (noncoinciding) occurrences $b_1$ and $b_3$. Let $F'$ be a subformula of $F$ which does not contain one of the three given occurrences and contains another not on the left. Then $b_2$ is a left occurrence in the subformula $F'$.*

**Corollary 2.** *Let $F_1$ and $F_2$ be disjoint subformulas of the formula $F$. If two occurrences belong to these two subformulas and are adjacent, then they are left occurrences of the subformulas $F_1$ and $F_2$.*

**Lemma 7.** *Let $b_1$ and $b_2$ be noncoinciding occurrences in a formula $F$. Then there exists a sequence of pairwise distinct occurrences $\bar{b}_1, \bar{b}_2, \dots, \bar{b}_m$, $m \ge 2$, such that $\bar{b}_1$ is $b_1$, $\bar{b}_m$ is $b_2$, and for $i = 1, 2, \dots, m-1$, $\bar{b}_i$ is adjacent to $\bar{b}_{i+1}$.*

**4. Proof of the main lemma.** The subformula $\Phi'$ of the formula $\Phi$ is decomposed into disjoint subformulas of the form $\varphi_i$, which are the result of substitution in place of all occurrences in $f$. We shall call these subformulas elementary.

Similarly, decompose the subformula $\Psi'$ of the formula $\Psi$ into elementary subformulas. By Lemma 1, $f$ and $g$ contain the same variables; therefore $\Phi'$ and $\Psi'$ contain the same elementary subformulas. From the definition of adjacency it follows that the left occurrences of two elementary subformulas in $\Phi'$ ($\Psi'$) are adjacent if and only if in $f$ ($g$) the occurrences in whose place the given elementary subformulas are substituted are adjacent. Further, in $f$ there are no variables adjacent to themselves: otherwise such a variable would be found in $\Phi$. Therefore, by Lemma 3, the formulas $f$ and $g$ have identical left variables and identical sets of pairs of adjacent variables. Hence, in particular, it follows that the left elementary subformulas in $\Phi'$ and $\Psi'$ are the same. We shall call the left occurrences of elementary subformulas supporting.

By the hypothesis of the lemma, in $\Phi$ there exists an occurrence $b_2$ of the variable $x_2$, adjacent to an occurrence $b_1$ of the variable $x_1$ and to an occurrence $b_3$ of the variable $x_3$. Obviously, the following three cases exhaust all possibilities:

1. Each of $b_1, b_2, b_3$ lies outside $\Phi'$ or is a left occurrence in $\Phi'$.

II. $b_1, b_2, b_3$ lie in $\Phi'$.

III. Among $b_1, b_2, b_3$ at least one lies outside $\Phi'$ and at least one is not a left occurrence in $\Phi'$.

**Case I.** From the definition of adjacency it follows that if two occurrences outside $\Phi'$ are adjacent, then after replacing $\Phi'$ by $\Psi'$ these occurrences are adjacent in $\Psi$, and that if the left occurrence of the subformula $\Phi'$ is adjacent to an occurrence outside $\Phi'$, then in $\Psi$ the latter is adjacent to the left occurrence of the subformula $\Psi'$. To complete the proof it remains to note that the left variables of $\Phi'$ and $\Psi'$ coincide.

**Case II.** If $b_1, b_2, b_3$ lie in one elementary subformula, then everything is proved, since the same subformula exists in $\Psi'$. Suppose that among $b_1, b_2, b_3$ there is a non-pivotal occurrence, but not all three occurrences belong to one elementary subformula. By the corollaries of Lemma 6, in this case $b_2$ is a pivotal occurrence, and of $b_1$ and $b_3$ one is pivotal while the other belongs to the same elementary subformula as $b_2$. Suppose, for example, that $b_1$ lies in the elementary subformula $\varphi_{i_1}$, and $b_2$ and $b_3$ in $\varphi_{i_2}$. The variables $y_{i_1}$ and $y_{i_2}$ are adjacent in $f$, and therefore in $g$ there are adjacent occurrences of the variables $y_{i_1}$ and $y_{i_2}$. Under substitution from them one obtains elementary subformulas in $\Psi'$, whose left occurrences are adjacent. Thus the left occurrence $x_2$ in the subformula $\varphi_{i_2}$ under consideration is adjacent in $\Psi'$ to the left occurrence $x_1$ in $\varphi_{i_1}$; moreover, in $\varphi_{i_2}$ the left occurrence $x_2$ is adjacent to $x_3$.

Thus, in case II it remains to consider the subcase when $b_1, b_2, b_3$ are pivotal occurrences. In this case in $\Psi'$ there are also pivotal occurrences $\bar{b}_1, \bar{b}_2, \bar{b}_3$ of the variables $x_1, x_2, x_3$. Using Lemma 7, construct a sequence

$$\bar{b}_1, b'_1, b'_2, \dots, b'_l, \bar{b}_3, \quad l \geq 0,$$

of pairwise distinct occurrences in $\Psi'$, in which any two neighboring occurrences are adjacent. All these occurrences are pivotal: if among them there were a non-left occurrence of some elementary subformula, then the left occurrence of the same subformula would occur in the sequence of occurrences under consideration twice, since, by Lemma 6, one can "enter" a subformula and "leave" it only through the left occurrence. Let

$$P = x_1 x_{i_1} x_{i_2} \dots x_{i_l} x_3$$

be the corresponding sequence of variables. From the conditions of the main lemma it follows that Lemma 5 is applicable to $P$. By this lemma, $P$ contains the subword $x_1 x_2 x_3$; in the original sequence of occurrences this subword corresponds to the desired triple of occurrences of the variables $x_1, x_2, x_3$.

**Case III.** From Corollary 1 of Lemma 6 it follows that in this case $b_2$ is a left occurrence in $\Phi'$. Suppose, for example, that $b_3$ lies outside $\Phi'$, and $b_1$ in $\Phi'$. If $b_1$ belongs to a left elementary subformula, then, since the left elementary subformula in $\Psi'$ has the same form, the lemma is proved. Otherwise, by Corollary 2 of Lemma 6, $b_1$ is a pivotal occurrence. Hence, in $\Psi'$ there is a pivotal occurrence $\bar{b}_1$ of the variable $x_1$. Connect $\bar{b}_1$ with the left occurrence $\bar{b}_2$ of the variable $x_2$ in $\Psi'$ by a sequence of occurrences in which neighboring occurrences are adjacent. As before, all occurrences in it are pivotal. $\bar{b}_2$ is adjacent in this sequence either to an occurrence of the variable $x_1$, or to an occurrence of $x_3$ (for $\Psi \in K_n$). In the first case everything is proved, since $\bar{b}_2$ is adjacent to $b_3$, "inherited" from $\Phi$; in the second case in $\Psi'$ there are pivotal occurrences $x_1, x_2, x_3$. For this case the lemma has already been proved in the analysis of case II. Thus, the main lemma is completely proved. Hence the theorem is completely proved.

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*Note: Figure translations are in progress. See original paper for figures.*