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Abstract

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COUPLED OSCILLATORS

IN THE ADIABATIC APPROXIMATION

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PHYSICS

It is well known that in a system with slowly varying parameters there may exist a quantity conserved in time with exponential accuracy—an adiabatic invariant. The validity of the latter is easily established by means of the method of stationary phase (^{1}), and the problem is, in essence, completely equivalent to the problem of over-barrier reflection in quantum mechanics in the quasiclassical approximation (^{2}). In a system consisting of several coupled oscillators, the question of the conservation of adiabatic invariants is immediately complicated by the possibility that the oscillators pass through resonance (the "resonance point" may then lie in the complex plane of the time t). If, for example, at $t \to -\infty$ some one oscillator was excited, then at $t \to +\infty$ the most varied redistributions of the initial energy among the oscillators are possible. In the present paper a system of two coupled oscillators whose parameters vary slowly with time is investigated in detail. The formal aspect of the question consists in constructing an asymptotic solution of a fourth-order equation under prescribed boundary conditions.

The Lagrangian function of the system under investigation has the form

$$L = {}^1/_2(\dot{x}^2 + \dot{y}^2) - {}^1/_2\omega_1^2(t)x^2 - {}^1/_2\omega_2^2(t)y^2 + a(t)xy, \tag{1}$$

where it will be assumed in what follows that the coupling parameter $a < \omega_1\omega_2$ (the case when $a > \omega_1\omega_2$, as will be seen below, is of no physical interest, and its consideration is trivial). The equations of motion have the form

$$\ddot{x} + \omega_1^2(t)x = a(t)y, \qquad \ddot{y} + \omega_2^2(t)y = a(t)x. \tag{2}$$

The natural frequencies $\omega_1(t)$, $\omega_2(t)$ and the coupling parameter $a(t)$ are assumed to be analytic functions and nowhere vanish on the real axis t. If the parameters ω_1, ω_2, a did not depend on t, it would be possible to reduce the sum of the two quadratic forms (1) to canonical form by a single transformation. In the case under consideration such a transformation cannot be carried out in a uniform way over the entire t-axis. This is connected with the fact that the transformation matrix, now depending on t through the parameters ω_1, ω_2, a, is singular at points in the plane of the complex variable t where the characteristic numbers of the quadratic form ${}^1/_2(\omega_1^2 x^2 + \omega_2^2 y^2 - axy)$ coincide ([3]). System (2) was studied in ([4]) in connection with the problem of crossing of terms in atomic collisions. The results of ([4]) were used in ([5]) for problems on the transformation of waves in plasma. The investigations carried out in ([4], [5]) are, in a certain sense, inexact (the corresponding remarks will be made below).

For what follows it is convenient to reduce system (2) to the form

$$\beta \frac{d^2 x}{d\tau^2} + \Omega_1^2(\tau) x = \Gamma y, \qquad \beta \frac{d^2 y}{d\tau^2} + \Omega_2^2(\tau) y = \Gamma x \quad (\beta \ll 1), \tag{3}$$

where $\tau = t/T$; T is the characteristic time of variation of the parameters*; $\Omega_1 \sim \Omega_2 \sim 1$; the quantity $\Gamma \sim a/\omega^2$ characterizes the degree of coupling of the oscillators. The solution of system (3) is sought in the form of the expansion:

$$x, y = \exp\left\{ \frac{i}{\sqrt{\beta}} \int^{\tau} \left(\bar{k}_0(\tau) + \sqrt{\beta}\, \bar{k}_1(\tau) + \cdots \right) d\tau \right\}. \tag{4}$$

The principal terms of the four asymptotic solutions are determined by the expressions:

$$x_{\pm} = \Pi_1 \exp\left[\pm i \int^t k_1(\tau)\, d\tau \right], \qquad y_{\pm} = \Pi_2 \exp\left[\pm i \int^t k_2(\tau)\, d\tau \right], \tag{5}$$

which are obtained after substituting (4) into (3). Here

$$\pm k_{1,2} = \pm \sqrt{ \frac{\Omega_1^2 + \Omega_2^2}{2} \mp \sqrt{ \frac{(\Omega_1^2 - \Omega_2^2)^2}{4} + \Gamma^2 } }; \tag{6}$$

$\Pi_{1,2}$ are pre-exponential factors determined in (4).

The further problem consists in the following: having specified, for example, at $+\infty$ the solution Z in the form

$$Z_+ = A_0 x_+ + B_0 y_+ + C_0 y_- + D_0 x_-, \tag{7}$$

we must determine the solution Z_- at $-\infty$. The singular points of the asymptotic solution Z are, as already noted, the points of coincidence of the characteristic roots $\pm k_{1,2}$. In this case the following four cases are possible: 1) $k_1 = -k_1 = 0$; 2) $k_2 = -k_2 = 0$; 3) $k_1 = k_2$; 4) $k_1 = -k_2$. All these points, by virtue of the restriction imposed earlier on ω_1, ω_2, a, do not lie on the real t-axis.

Fig. 1

Figure 1: Fig. 1

The first two cases correspond to the usual turning points in the Schrödinger equation, and their consideration is known (see, for example, ([6])). In works ([4],[5]) only case 3 was considered. It should be noted that the root k of the characteristic equation has 4 branches and, consequently, generally speaking, there are 6 possible cases of intersection of roots. Since the equation for determining k is biquadratic, the indicated 4 cases remain, and ignoring any of them is, generally speaking, unlawful.

Fig. 1

Let us represent the solutions (5) in the form:

$$\begin{aligned} x_{\pm} &= \Pi_1 \exp\left[\pm i \int^t \frac{k_1+k_2}{2}\,dt\right] \exp\left[\pm i \int^t \frac{k_1-k_2}{2}\,dt\right], \\ y_{\pm} &= \Pi_2 \exp\left[\pm i \int^t \frac{k_1+k_2}{2}\,dt\right] \exp\left[\mp i \int^t \frac{k_1-k_2}{2}\,dt\right] \end{aligned} \tag{8}$$

and restrict ourselves to the consideration of the case when the expressions $(k_1-k_2)^2$ and $(k_1+k_2)^2$ have simple zeros respectively at the points O_1, O_2 and O_1', O_2' (Fig. 1). By virtue of the reality of the coefficients of the characteristic equation, the roots $t_{O_1}, t_{O_1'}$ and $t_{O_2}, t_{O_2'}$ are respectively complex conjugates. The coefficients A_0, B_0, C_0, D_0 in (7) change discontinuously upon tran–

* For simplicity we take T to be the same for ω_1, ω_2, a, and $\omega_1 \sim \omega_2$. As will be seen from what follows, this does not restrict the generality of the consideration.

along one level line, where $\mathrm{Im}(k_1 \pm k_2) = 0$ (Stokes lines), to another. In this process the lines $1, 2, 3, 4, 1', 2', 3', 4'$ are crossed, on which $\mathrm{Re}(k_1 \pm k_2) = 0$. When the points O_1, O_2 are encircled, the pairs of solutions (x_+, y_+) and (x_-, y_-) behave independently; when the points O_1', O_2' are encircled, the pairs (x_+, y_-) and (x_-, y_+) exhibit the same independence. On the lines $1, 3$, $\mathrm{Im}(k_1 - k_2) < 0$; on the lines $2, 4$, $\mathrm{Im}(k_1 - k_2) > 0$; on the lines $1', 3'$, $\mathrm{Im}(k_1 + k_2) > 0$; on the lines $2', 4'$, $\mathrm{Im}(k_1 + k_2) < 0$.

To determine the connection rules for the solutions (5) in the presence of the special points $k_1 = \pm k_2$, we shall use a method analogous to Zwaan' s method.

Taking into account the preceding considerations, we shall encircle the points O_1, O_2, O_1', O_2' along the contour shown in Fig. 1. Starting from the point P_1 with the solution in the form (7), we shall put the index i on the coefficients after crossing the line with number i (when passing through primed lines we shall mark the coefficients with primes). We have:

$$A_1 = M_1 A_0; \quad B_1 = B_0/M_1 + \alpha_1 M_1 A_0; \quad C_1 = M_2 C_0; \quad D_1 = D_0/M_2 + \alpha_1 M_2 C_0;$$

$$A_2 = A_1 + \beta_1 B_1; \quad B_2 = B_1; \quad C_2 = C_1 + \beta_1 D_1; \quad D_2 = D_1;$$

$$A'_4 = M_1 N_1 A_2; \quad B'_4 = \frac{N_2}{M_1} B_2; \quad C'_4 = \frac{M_2}{N_1} C_2 + \alpha_2 M_1 N_1 A_2;$$

$$D'_4 = \frac{D_2}{M_2 N_2} + \alpha_2 \frac{N_2}{M_1} B_2;$$

$$A'_3 = A'_4 + \beta_2 C'_4; \quad B'_3 = B'_4 + \beta_2 D'_4; \quad C'_3 = C'_4; \quad D'_3 = D'_4; \tag{9}$$

$$A'_2 = N_1^2 A'_3; \quad B'_2 = N_2^2 B'_3; \quad C'_2 = \frac{C'_3}{N_1^2} + \gamma_2 A'_3 N_1^2; \quad D'_2 = \frac{D'_3}{N_2^2} + \gamma_2 N_2^2 B'_3;$$

$$A_3 = N_1 M_1 A'_1; \quad B_3 = \frac{N_2}{M_1} B'_1 + \gamma_1 N_1 M_1 A'_1; \quad C_3 = \frac{M_2}{M_1} C'_1;$$

$$D_3 = \frac{D'_1}{M_2 N_2} + \gamma_1 \frac{M_2}{N_1} C'_1;$$

$$A'_1 = A'_2 + \delta_2 C'_2; \quad B'_1 = B'_2 + \delta_2 D'_2; \quad C'_1 = C'_2; \quad D'_1 = D'_2;$$

$$A_4 = A_3 + \delta_1 B_3; \quad B_4 = B_3; \quad C_4 = C_3 + \delta_1 D_3; \quad D_4 = D_3,$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ are undetermined factors;

$$M_{1,2} = \exp\left\{\frac{l_1 \pm i\varphi_1}{2}\right\};$$

$$N_{1,2} = \exp\left\{\frac{l_2 \pm i\varphi_2}{2}\right\};$$

$$l_{1,2} = -\frac{i}{4\sqrt{\beta}} \int_{L_{1,2}} (k_1 \mp k_2)\, d\tau > 0; \tag{10}$$

$$L_1 = P_1O_1P_2O_2P_1; \quad L_2 = P_1'O_2'P_2'O_1'P_1';$$

$\varphi_{1,2}$ are undetermined phase advances arising from the pre-exponential factor. Returning to the point P_1 and requiring agreement with (7), by virtue of analyticity of the solution we obtain, taking into account the uniqueness of the solution,

$$\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = i\sqrt{1 - e^{-2l_1}}; \quad \alpha_2 = \beta_2 = \gamma_2 = \delta_2 = i\sqrt{1 - e^{-2l_2}};$$

$$\varphi_1 = \varphi_2 = \pi/4. \tag{11}$$

Formulas (9)–(11) solve the problem of matching the asymptotic solutions (5). Hence we readily find

$$\begin{aligned}
A_0' &= e^{-l_1-l_2}A_0 + e^{-l_2}\sqrt{1-e^{-2l_1}}\,B_0 - e^{-l_1}\sqrt{1-e^{-2l_2}}\,C_0 + \\
&\quad + \sqrt{(1-e^{-2l_1})(1-e^{-2l_2})}\,D_0; \\
B_0' &= -e^{-l_2}\sqrt{1-e^{-2l_1}}\,A_0 + e^{-l_1-l_2}B_0 + \\
&\quad + \sqrt{(1-e^{-2l_1})(1-e^{-2l_2})}\,C_0 + e^{-l_1}\sqrt{1-e^{-2l_2}}\,D_0; \\
C_0' &= e^{-l_1}\sqrt{1-e^{-2l_2}}\,A_0 + \sqrt{(1-e^{-2l_1})(1-e^{-2l_2})}\,B_0 + \\
&\quad + e^{-l_1-l_2}C_0 - e^{-l_2}\sqrt{1-e^{-2l_1}}\,D_0; \\
D_0' &= \sqrt{(1-e^{-2l_1})(1-e^{-2l_2})}\,A_0 - e^{-l_1}\sqrt{1-e^{-2l_2}}\,B_0 + \\
&\quad + e^{-l_2}\sqrt{1-e^{-2l_1}}\,C_0 + e^{-l_1-l_2}D_0,
\end{aligned} \tag{12}$$

where it has been taken into account that, as a result of a half-circuit, the replacements $x_\pm \to x_\mp$ and $y_\pm \to y_\mp$ occur.

Formulas (12) completely solve the question of the distribution of energy among the various degrees of freedom of two coupled oscillators for given initial conditions.

The result obtained can be formulated as the following theorem: in a system of two coupled oscillators with slowly time-varying parameters $(\omega_1, \omega_2, \alpha)$, the "internal resonances" $(k_1 = \pm k_2)$ leave invariant the action of the system

$$I = I_x + I_y; \qquad I_x = \frac{E_x}{k_1}; \qquad I_y = \frac{E_y}{k_2}, \tag{13}$$

where $E_{x,y}$ is the energy of the corresponding oscillator.

Indeed, it follows from (12) that the transformation from (A_0, B_0, C_0, D_0) to (A_0', B_0', C_0', D_0') is unitary and leaves invariant the quantity $|A_0|^2 + |B_0|^2 = |A_0'|^2 + |B_0'|^2$ $(C_0 = B_0^*;\ D_0 = A_0^*)$. Taking into account that

$$\Pi_{1,2} = 1/\sqrt{k_{1,2}}; \qquad A_0 = \bar{A}\sqrt{k_1}; \qquad B_0 = \bar{B}\sqrt{k_2}, \tag{14}$$

where $\bar{A}, \bar{B}$ are the amplitudes of the oscillations, we immediately arrive at (13) $(E_x = |\bar{A}|^2 k_1^2,\ E_y = |\bar{B}|^2 k_2^2)$.

It is significant that the actions of the subsystems $I_x,\ I_y$ need not be conserved. Between different degrees of freedom there may occur an intensive exchange of energy, depending on the frequencies $k_1,\ k_2$ and on the parameters $l_1,\ l_2$. The magnitude of the latter is determined by the ratio $\Gamma/\sqrt{\dot{\beta}}$.

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Note: Figure translations are in progress. See original paper for figures.