ON CERTAIN ESTIMATES OF THE COEFFICIENTS OF BOUNDED HOLOMORPHIC FUNCTIONS

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Abstract

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MATHEMATICS

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ON CERTAIN ESTIMATES OF THE COEFFICIENTS OF BOUNDED HOLOMORPHIC FUNCTIONS

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1. In the article ([1]) by I. A. Aleksandrov, in the case of a complete circular domain $D \ni (0, \dots, 0)$ of the space of n complex variables, sharp estimates were obtained for the Taylor coefficients of functions: a) holomorphic and bounded in D; b) holomorphic in D and having there a positive real part, generalizing the known estimates for these functions in the case of one variable in the disk.

Considerably earlier, in the case of n complex variables, the problem of obtaining sharp estimates of the Taylor coefficients for functions

$$F(z_1, \dots, z_n) = \sum_{k_1, \dots, k_n = 0}^{\infty} c_{k_1 \dots k_n} z_1^{k_1} \cdots z_n^{k_n},$$

belonging to the functions of Schur and Carathéodory, was considered by the author in the note ([2]), of which I. A. Aleksandrov was apparently unaware. Indeed, the estimates found for the indicated functions in ([1]) in the case of the polydisc $\overline{L}\{|z_m| \leqslant r_m \ (m = 1, \dots, n)\}$ for $\|k\| > 0$, namely $|c_k| \leqslant (1 - |c_0|^2) r^{-k}$ (see ([1]), p. 8), $|c_k| \leqslant 2r^{-k} \operatorname{Re} c_0$ (see ([1]), p. 5) ($\|k\| = k_1 + \dots + k_n,\ c_k = c_{k_1 \dots k_n}$ and $r^{-k} = r_1^{-k_1} \cdots r_n^{-k_n}$), of which, as an immediate consequence in the case of the domain D, the corresponding estimates are (see ([1]), theorems 2, 1)

$$|c_k| \leqslant (1 - |c_0|^2)/\Delta_k(D) \quad \left(\Delta_k(D) = \Delta_{k_1 \dots k_n}(D) = \sup_{(z_1, \dots, z_n) \in D} (|z_1|^{k_1} \cdots |z_n|^{k_n}) \right)^*, \tag{1}$$

$$|c_k| \leqslant (2/\Delta_k(D)) \operatorname{Re} c_0, \tag{2}$$

were obtained by the author (see ([2]), lemmas 1 and 2**) as early as 1960 in the case of the polydisc $S\{|z_m| < R_m (m = 1, \dots, n)\}$ (and hence, a fortiori, in the case $\overline{L}$). On the basis of the author' s results in the case S, in [2], by the same method as that used by I. A. Aleksandrov ([1]), sharp estimates of the coefficients of Schur and Carathéodory functions were obtained also in the case of a relatively broad special class of complete circular domains of the space of n complex variables. The latter estimates, in contrast to estimates (1), (2), if one does not count the simplest domains D ([1]), are effective, since in them, owing to the indicated selected class of domains,

$$\sup_{(z_1,\dots,z_n)\in D}(|z_1|^{k_1}\cdots|z_n|^{k_n})$$

is effectively computed ([2])***.

Therefore, in the case of the number of complex variables $n>1$, the author usually confined himself to considering special classes of complete circular–

* In estimate (1) the author in ([3,4]) clarified the question of uniqueness of extremal functions, which is not es

** In lemma 2 the function $F(z_1,\dots,z_n)$ is normalized by the condition $F(0,\dots,0)=4,$ which, of course, enta

*** In ([4]) an even broader class of domains D is singled out, for which $\sup_{(z_1,\dots,z_n)\in D}(|z_1|^{k_1}\cdots|z_n|^{k_n})$ is effective

domains ([3], [5–7]), although (if one abandons the effectiveness of the coefficient estimates) it is clear from the preceding that, in the case of S, estimates (1), (2) immediately follow from the author' s results mentioned above, as does (4), which was noted in 1964. We note that in the case $n=2$ an important role is played by the special class of complete bicircular domains singled out earlier by A. A. Temlyakov ([8]), which also makes it possible to obtain effective coefficient estimates ([3], [5–7]).

2. Let $f(z)$ be regular, nonzero, and bounded in the disk $|z|<R$, $|f(z)|\le M$. If $p\ge 1$ is the least integer for which $f^{(p)}(0)\ne 0$, then, as Rajagopal ([9]) proved, the estimate

$$(k!)^{-1}|f^{(k)}(0)|\le 2R^{-k}|f(0)|\ln(|f(0)|^{-1}M),\qquad p\le k\le 2p-1. \tag{3}$$

holds.

A consequence of estimate (3) is the estimate ([9])

$$(k!)^{-1}|f^{(k)}(0)|\le 2M(eR^k)^{-1},\qquad p\le k\le 2p-1. \tag{4}$$

Later, in the case of two complex variables, an analogous proposition was established by the author ([6]).

In ([1]), for Schur functions there are the following two propositions (see ([1]), Theorems 3 and 4), adjacent to the indicated proposition of Rajagopal.

I. If the function

$$F(z_1, \dots, z_n) = \sum_{k_1,\dots,k_n=0}^{\infty} c_{k_1\dots k_n} z_1^{k_1} \cdots z_n^{k_n}$$

holomorphic in the domain D is bounded in D, $|F(z_1, \dots, z_n)| < M$, and does not assume the value zero in D, then

$$|c_{k_1\dots k_n}| \le 2(|c_{0\dots 0}|/\Delta_{k_1\dots k_n}(D)) \ln(|c_{0\dots 0}|^{-1} M), \qquad k_1 + \cdots + k_n > 0. \tag{5}$$

II. If the function

$$f(z) = \sum_{k=l}^{\infty} c_k z^k$$

holomorphic in the disk $|z| < R$, with $c_l \neq 0$ $(l = 0, 1, \dots)$, is nonzero in $0 < |z| < R$ and bounded in $|z| < R$, $|f(z)| < M$, then

$$|c_k| \le 2R^{l-k} |c_l| \ln(|c_l|^{-1} R^{-l} M), \qquad k > l, \tag{6}$$

whence

$$|c_k| \le 2M(eR^k)^{-1}, \qquad k > l. \tag{7}$$

In the proof of Proposition I, in the equality (see ([1]), p. 10)

$$(k_1! \cdots k_n!)^{-1} \Phi^{(k_1+\cdots+k_n)}_{z_1^{k_1} \dots z_n^{k_n}}(0, \dots, 0) = c_{k_1\dots k_n}/c_{0\dots 0}, \qquad k_1 + \cdots + k_n > 0, \tag{8}$$

where $\Phi(z_1, \dots, z_n) = \ln F(z_1, \dots, z_n)$, an oversight was made. It is evident that, under the conditions of Proposition I, (8) holds for $k_1 + \cdots + k_n = 1$; for $k_1 + \cdots + k_n > 1$ it need not necessarily be satisfied. An oversight of the same character was also made in the proof of Proposition II (see ([1]), p. 10). In this connection, Propositions I and II require clarification, which is carried out below.

By virtue of the preceding, under the conditions of Proposition I, estimate (5) is proved for $k_1 + \dots + k_n = 1$. Similarly, under the conditions of Proposition II, estimate (6) is proved for $k = l + 1$. However, for $k_1 + \dots + k_n > 1$, estimate (5) in Proposition I, and for $k > l + 1$, estimate (6) in Proposition II, do not hold*.

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* Indeed, for the function (for brevity of notation, as D we take the unit polydisk $S_1\{|z_m| < 1\ (m = 1, \dots, n)\}$)

$$F(z_1, \dots, z_n) = \exp\left[\alpha z_1 + \dots + \alpha z_n + (1 + z_1^{k_1} \cdots z_n^{k_n})^{-1} z_1^{k_1} \cdots z_n^{k_n}\right],$$

where α is a positive constant satisfying

$$\alpha > (k_1! \cdots k_n!\, 2n)^{(k_1+\dots+k_n-1)^{-1}}, \qquad k_1 + \dots + k_n > 1, \tag{A}$$

we have

$$(k_1! \cdots k_n!)^{-1} \left| F^{(k_1+\dots+k_n)}_{z_1^{k_1} \dots z_n^{k_n}}(0, \dots, 0) \right| = 1 + (k_1! \cdots k_n!)^{-1} \alpha^{k_1+\dots+k_n}.$$

At the same time the following correction of the estimates (5), (6), respectively in assertions I, II, is possible. If relation (8) is excluded from consideration, then, as is easy to see, the proof of assertion I (see (1), p. 10) leads to the estimate

$$(k_1! \dots k_n!)^{-1} \left| \Phi^{(k_1+\dots+k_n)}_{z_1^{k_1} \dots z_n^{k_n}}(0, \dots, 0) \right| \leq (2/\Delta_{k_1 \dots k_n}(D)) \ln(|c_{0\dots0}|^{-1} M),$$

$k_1 + \dots + k_n > 0$, where $\Phi(z_1, \dots, z_n)$ is the function indicated above. Therefore, putting

$$(k_1! \dots k_n!)^{-1} \Phi^{(k_1+\dots+k_n)}_{z_1^{k_1} \dots z_n^{k_n}}(0, \dots, 0) = a_{k_1 \dots k_n},$$

under the condition of assertion I, instead of estimate (5) we shall have the estimate

$$|a_{k_1 \dots k_n}| \leq (2/\Delta_{k_1 \dots k_n}(D)) \ln(|c_{0\dots0}|^{-1} M), \qquad k_1 + \dots + k_n > 0. \tag{9}$$

Analogously, under the condition of assertion II, instead of estimate (6) the estimate

$$|a_{k-l}| \leq 2R^{l-k} \ln(|c_l|^{-1} R^{-l} M), \qquad k > l, \tag{10}$$

will be valid, where

$$a_{k-l} = [(k-l)!]^{-1}\Phi^{(k-l)}(0) \quad (\Phi(z) = \ln(f(z)/z^l),\ \Phi(0) = \ln c_l).$$

Naturally, in this case the estimate (7)—a consequence of estimate (6)—must be discarded.

Remark 1. The estimates (9), (10) are sharp, since they are attained respectively by the functions

$$F(z_1, \dots, z_n) = \exp\left[(\Delta_{k_1 \dots k_n}(D) + z_1^{k_1} \dots z_n^{k_n})^{-1} z_1^{k_1} \dots z_n^{k_n}\right]; \tag{11}$$

$$f(z) = z^l \exp\left[(R^{k-l} + z^{k-l})^{-1} z^{k-l}\right], \qquad k > l.$$

If, in addition, there is an extra condition in addition to the condition of assertion I, we shall have the following extension of Rajagopal' s assertion to the case of n complex variables.

Theorem 1. Let in the domain D the function

$$F(z_1, \dots, z_n) = \sum_{k_1, \dots, k_n = 0}^{\infty} c_{k_1 \dots k_n} z_1^{k_1} \dots z_n^{k_n}$$

be regular, different from zero and bounded,

$$|F(z_1, \dots, z_n)| \leq M.$$

If $p \geq 1$ is the least natural number for which the sum of the moduli of all coefficients $c_{k_1 \dots k_n}$ of order p* is not equal to zero, then

$$|c_{k_1 \dots k_n}| \leq 2\,(|c_{0 \dots 0}|/\Delta_{k_1 \dots k_n}(D)) \ln(|c_{0 \dots 0}|^{-1} M), \tag{12}$$

$$p \leq k_1 + \dots + k_n \leq 2p - 1.$$

Proof. By the condition of the theorem, the function

$$\Phi(z_1, \dots, z_n) = \ln F(z_1, \dots, z_n)$$

is regular in D and satisfies there the condition

$$\operatorname{Re}\Phi(z_1, \dots, z_n) \leq \ln M,$$

and moreover

$$(k_1! \dots k_n!)^{-1} \Phi^{(k_1+\dots+k_n)}_{z_1^{k_1} \dots z_n^{k_n}}(0, \dots, 0) =$$

According to estimate (5)

$$(k_1! \dots k_n!)^{-1} \left| F^{(k_1+\dots+k_n)}_{z_1^{k_1} \dots z_n^{k_n}}(0, \dots, 0) \right| \leq 1 + 2\alpha n$$

(for S_1. $\Delta_{k_1 \dots k_n}(D) = 1$, $|F(z_1, \dots, z_n)| < \exp(\alpha n + 2^{-1}) = M$). But, by virtue of inequality (A), for $k_1 + \dots + k_n > 1$

$$1 + (k_1! \dots k_n!)^{-1} \alpha^{k_1+\dots+k_n} > 1 + 2\alpha n.$$

We arrive at an analogous conclusion also in the case of the function satisfying the condition of assertion II,

$$f(z) = R^{-l} z^l \exp\left[R^{-1}\alpha z + (R^{k-l} + z^{k-l})^{-1} z^{k-l}\right] \quad (k > l + 1),$$

where α is a positive constant satisfying

$$\alpha > [(k-l)!\,2]^{(k-l-1)^{-1}}.$$

* As is known, the order of the coefficient $c_{k_1 \dots k_n}$ is the number $k = k_1 + \dots + k_n$.

$$= c_{k_1 \dots k_n} / c_{0 \dots 0}, \quad p \leqslant k_1 + \dots + k_n \leqslant 2p - 1.$$

Therefore, according to the well-known proposition*, applied to the function $\Phi(z_1, \dots, z_n)$, we have

$$|c_{0 \dots 0}|^{-1} |c_{k_1 \dots k_n}| \leqslant (2/\Delta_{k_1 \dots k_n}(D))(\ln M - \ln |c_{0 \dots 0}|),$$

$$p \leqslant k_1 + \dots + k_n \leqslant 2p - 1.$$

This gives the estimate (12).

Corollary 1. *Under the hypotheses of Theorem 1 we have the estimate*

$$|c_{k_1 \dots k_n}| \leqslant 2e^{-1} M / \Delta_{k_1 \dots k_n}(D), \quad p \leqslant k_1 + \dots + k_n \leqslant 2p - 1. \tag{13}$$

Remark 2. The estimates (12), (13), for $k_1 + \dots + k_n = p$, are sharp, since they are attained respectively by the functions (11),

$$F(z_1, \dots, z_n) = \exp\left[(z_1^{k_1} \cdots z_n^{k_n} + \Delta_{k_1 \dots k_n}(D))^{-1} (z_1^{k_1} \cdots z_n^{k_n} - \Delta_{k_1 \dots k_n}(D))\right],$$

for which $k_1 + \dots + k_n = p$.

Introducing an additional condition into the hypothesis of Proposition II, we obtain the following generalization of Rajagopal' s proposition.

Theorem 2. *Let the function*

$$f(z) = \sum_{k=l}^{\infty} c_k z^k, \quad c_l = 0 \ (l = 0, 1, \dots),$$

regular in the disk $|z| < R$, be different from zero in $0 < |z| < R$, bounded in $|z| < R$, $|f(z)| \leqslant M$, and let $p \geqslant l + 1$ be the smallest index for which $c_p \neq 0$. Then

$$|c_k| \leqslant 2R^{l-k}|c_l| \ln(|c_l|^{-1} R^{-l} M), \quad p \leqslant k \leqslant 2p - l - 1. \tag{14}$$

Proof. The function $\varphi(z) = f(z)/z^l$, $\varphi(0) = c_l$, is regular in the disk $|z| < R$, is different from zero, is bounded, $|\varphi(z)| \leqslant MR^{-l}$, and has the expansion

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = c_{l+n},$$

where $q = p - l \geqslant 1$ is the smallest index for which $a_q \neq 0$. Consequently, according to estimate (3), we have

$$|a_n| \leqslant 2R^{-n}|a_0| \ln(|a_0|^{-1} R^{-l} M), \quad p - l \leqslant n \leqslant 2(p - l) - 1.$$

This gives the estimate (14).

Corollary 2. *Under the hypotheses of Theorem 2 we have the estimate*

$$|c_k| \leqslant 2M(eR^k)^{-1}, \quad p \leqslant k \leqslant 2p - l - 1. \tag{15}$$

Remark 3. The estimates (14), (15), for $k = p$, are sharp, since they are attained respectively by the functions

$$f(z) = R^{-l} z^l \exp\left[(z^{p-l} + R^{p-l})^{-1} z^{p-l}\right],$$

$$f(z) = R^{-l} z^l \exp\left[(z^{p-l} + R^{p-l})^{-1}(z^{p-l} - R^{p-l})\right], \quad p \geqslant l + 1.$$

The sharpness of estimate (4) for $k = p$ was established by Rajagopal (9).

Remark 4. Relying on Theorem 1, in a manner analogous to Theorem 2, one can prove a theorem extending Theorem 2 to the case of n complex variables.

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* If the function

$$F(z_1, \dots, z_n) = \sum_{k_1,\dots,k_n=0}^{\infty} c_{k_1 \dots k_n} z_1^{k_1} \cdots z_n^{k_n}$$

is holomorphic in the domain D and $\operatorname{Re} F(z_1, \dots, z_n) < u_0$ in D, then the inequalities

$$|c_{k_1 \dots k_n}| \leqslant 2(u_0 - \operatorname{Re} c_{0 \dots 0})/\Delta_{k_1 \dots k_n}(D), \quad k_1 + \dots + k_n > 0$$

hold (see (1), p. 6, Corollary 1). From the proofs of Theorem 1 (1) and of its Corollary 1 (1) it is clear that one may also have $\operatorname{Re} F(z_1, \dots, z_n) \leqslant u_0$. The same estimate was obtained independently in (4) (see (4), Theorem 4.6).

Note: Figure translations are in progress. See original paper for figures.