TORSION OF COMPOSITE ANISOTROPIC CYLINDRICAL BODIES WITH A SLIGHTLY CURVED AXIS

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Abstract

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T. M. KHATIASHVILI

TORSION OF COMPOSITE ANISOTROPIC CYLINDRICAL BODIES WITH A SLIGHTLY CURVED AXIS

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The problem of torsion of cylindrical bodies with a slightly curved axis in the case of an isotropic material was studied in works ([1,2]), and in the case of a transversely anisotropic material—in work ([3]).

1. Let a rectangular system of Cartesian coordinates $Ox_1x_2x_3$ be given. Consider an anisotropic cylindrical body possessing one plane of elastic symmetry perpendicular to the axis Ox_3. Then the generalized Hooke' s law may be written in the form

$$\begin{gathered} e_{ii} = E^{-1}(\sigma_{1i}\tau_{11} + \sigma_{2i}\tau_{22} + \sigma_{3i}\tau_{12} - \sigma_i\tau_{33}), \\ e_{12} = E^{-1}(\sigma_{13}\tau_{11} + \sigma_{23}\tau_{22} + \sigma_{33}\tau_{12} - \sigma_3\tau_{33}), \\ e_{33} = E^{-1}(\tau_{33} - \sigma_1\tau_{11} - \sigma_2\tau_{22} - \sigma_3\tau_{12}), \qquad e_{i3} = (-1)^i A_*^{-1}(A_{k5}\tau_{23} - A_{4k}\tau_{13}), \\ A_* = (A_{44}A_{55} - A_{45}) > 0 \qquad (i = 1, 2;\ k = 3 + i). \end{gathered} \tag{1,1}$$

Expressing from (1,1) the stress components τ_{ij} through the strain components e_{ij}, we obtain

$$\begin{gathered} \tau_{ii} = A_{1i}e_{11} + A_{2i}e_{22} + A_{3i}e_{33} + A_{6i}e_{12}, \\ \tau_{12} = A_{16}e_{11} + A_{26}e_{22} + A_{36}e_{33} + A_{66}e_{12}, \\ \tau_{13} = A_{55}e_{13} + A_{45}e_{23}, \qquad \tau_{23} = A_{45}e_{13} + A_{44}e_{23} \qquad (i = 1, 2, 3), \end{gathered} \tag{1,2}$$

where $E,\ \sigma_{ij}$ and A_{ij} are elastic constants ($\sigma_{ij} = \sigma_{ji},\ A_{ij} = A_{ji}$).

Suppose it is required to determine a particular solution of the equilibrium equations of an anisotropic elastic body in the presence of body forces:

$$\sum_{i=1}^{3} \frac{\partial \tau_{ij}}{\partial x_i} + a_j(x_1, x_2)x_3^k = 0 \qquad (j = 1, 2, 3), \tag{1,3}$$

where $a_j(x_1, x_2)$ are prescribed functions, and k is an integer nonnegative number. It is assumed that the corresponding strain components e_{ij}, determined by (1,1), satisfy the Saint-Venant compatibility conditions.

For $k = 0$, as a particular solution of equations (1,3) we may take the quantities:

$$\tau_{11}^{(0)} = \frac{\partial^2 \Psi_0}{\partial x_2^2} - \int a_1(x_1, x_2)\, dx_1, \qquad \tau_{22}^{(0)} = \frac{\partial^2 \Psi_0}{\partial x_1^2} - \int a_2(x_1, x_2)\, dx,$$
$$\tau_{12}^{(0)} = -\partial^2 \Psi_0 / \partial x_1 \partial x_2, \tag{1,4}$$
$$\tau_{33}^{(0)} = \sigma_1 \tau_{11}^{(0)} + \sigma_2 \tau_{22}^{(0)} + \sigma_3 \tau_{12}^{(0)}, \qquad \tau_{13}^{(0)} = A_{55} \partial \Phi_0 / \partial x_1 + A_{45} \partial \Phi_0 / \partial x_2,$$
$$\tau_{23}^{(0)} = A_{44} \partial \Phi_0 / \partial x_2 + A_{45} \partial \Phi_0 / \partial x_1.$$

In expressions (1,4), $\Phi_0(x_1, x_2)$ and $\Psi_0(x_1, x_2)$ are particular solutions, respectively, of the following equations:

$$\Delta_1 \Phi_0 = -a_3(x_1, x_2), \qquad \Delta_2 \Psi_0 = \beta_{11} \int \frac{\partial^2 a_1}{\partial x_2^2}\, dx_1 + \beta_{22} \int \frac{\partial^2 a_2}{\partial x_1^2}\, dx_2 +$$
$$+ \beta_{12} \left(\frac{\partial a_2}{\partial x_2} + \frac{\partial a_1}{\partial x_1} \right) - \beta_{13} \frac{\partial a_1}{\partial x_2} - \beta_{23} \frac{\partial a_2}{\partial x_1}, \tag{1,5}$$

where

$$\Delta_1 = A_{55} \partial_2 / \partial x_1^2 + 2A_{45} \partial^2 / \partial x_1 \partial x_2 + A_{44} \partial^2 / \partial x_2^2,$$
$$\Delta_2 = \beta_{22} \partial^4 / \partial x_1^4 + (2\beta_{12} + \beta_{33}) \partial^4 / \partial x_1^2 \partial x_2^2 - 2\beta_{13} \partial^4 / \partial x_1 \partial x_2^3 - 2\beta_{23} \partial^4 / \partial x_1^3 \partial x_2 + \beta_{11} \partial^4 / \partial x_2^4,$$
$$\beta_{ij} = E^{-1}(\sigma_{ij} - \sigma_i \sigma_j).$$

Using expressions (1,4) and the results of work ([5]), it is easy to construct a particular solution $\tau_{ij}^{(k)}$ of equations (1,3) for any k.

2. Consider an anisotropic cylindrical body t_0' with cavities, bounded externally by the planes $x_3 = 0$ and $x_3 = l$ $(l > 0)$, by the surface

$$f_{m+1}(x_1 + {}^1\!/_2 \varepsilon x_3^2,\ x_2) = 0, \tag{2,1}$$

and also by the surfaces of the cavities

$$f_j(x_1 + {}^1\!/_2 \varepsilon x_3^2,\ x_2) = 0 \quad (j = 1, 2, \ldots, m), \tag{2,2}$$

where ε is a small parameter whose squares and higher powers may be neglected.

Suppose that the cavities of the anisotropic body t'_0 are filled with other anisotropic materials, bonded to the body t'_0 along the surfaces (2,2). The bodies filling the cavities will be denoted by t'_j $(j = 1, 2, \ldots, m)$. Let all anisotropic materials have one plane of elastic symmetry, perpendicular to the axis Ox_3. The composite body formed by the bodies t'_j $(j = 0, 1, \ldots, m)$ will be denoted by t'. Make the substitution [1]

$$\xi_1 = x_1 + {}^1/_2 \varepsilon x_3^2, \qquad \xi_2 = x_2, \qquad \xi_3 = x_3. \tag{2,3}$$

Then the body t' is transformed into the cylindrical body t, and equations (2,1) and (2,2) take, respectively, the form

$$f_{m+1}(\xi_1, \xi_2) = 0, \tag{2,4}$$

$$f_j(\xi_1, \xi_2) = 0 \quad (j = 1, 2, \ldots, m). \tag{2,5}$$

To within terms of order ε^2, the relations [1] may be written as

$$\nu_i = \cos(n, \xi_i), \qquad \nu_3 = \varepsilon \xi_3 \cos(n, \xi_1); \qquad \partial/\partial x_i = \partial_i, \tag{2,6}$$

$$\partial/\partial x_3 = \partial_3 + \varepsilon \xi_3 \partial_1 \quad (i = 1, 2;\ \partial_j \equiv \partial/\partial \xi_j),$$

where ν_i are the direction cosines of the normal to the surfaces (2,1) and (2,2); n is the normal to the surfaces (2,4) and (2,5).

Let the origin of the coordinate system $O\xi_1\xi_2\xi_3$ be placed at the generalized center of inertia of one of the bases of the anisotropic body t, for example at $\xi_3 = 0$, and let the axes $O\xi_1$ and $O\xi_2$ be directed along the generalized principal axes of inertia of this base [4,5].

In Saint-Venant problems, the stress components τ_{ij} and the corresponding strain components e_{ij} must satisfy the equilibrium equations of the elastic body and the Saint-Venant compatibility conditions; the external forces on the surface (2,1) must be equal to zero, while the stress and displacement vectors arising on the elementary areas of the surfaces (2,2) must remain continuous when passing through these surfaces from the region t'_0 into the regions t'_j $(j = 1, 2, \ldots, m)$. The end conditions will be discussed below.

3. Suppose that the external forces applied to the base $x_3 = l$ of the composite anisotropic cylindrical body t' with a weakly curved axis are equivalent to a twisting couple with moment L. Taking account of the relations (2,3), we shall seek the solution of the problem in displacements as functions of ξ_i:

$$u_1^0 = -\tau\xi_2\xi_3 + \varepsilon\tau u_1, \qquad u_2^0 = \tau\xi_1\xi_3 + \varepsilon\tau u_2, \qquad u_3^0 = \tau\varphi(\xi_1, \xi_2) + \varepsilon\tau u_3, \ (3,1)$$

where u_i are unknown functions to be determined; ε is the small parameter indicated above; $\varphi(\xi_1, \xi_2)$ and τ are, respectively, the torsion function and the twist per unit length for the composite anisotropic cylindrical body t [(4)].

The stress components τ_{ij}^0, corresponding to the displacements (3,1), will have, to within terms of order ε^2, the form:

$$\tau_{jj}^0 = \varepsilon\tau(\tau_{jj} + A_{3j}\xi_3\partial_1\varphi), \qquad \tau_{12}^0 = \varepsilon\tau(\tau_{12} + A_{36}\xi_3\partial_1\varphi), \qquad (3,2)$$

$$\tau_{3i}^0 = \tau\left[\varepsilon\tau_{3i} + A_{5\gamma}(\partial_1\varphi - \xi_2) + A_{4\gamma}(\partial_2\varphi + \xi_1)\right]$$

$$(i = 1, 2; \quad j = 1, 2, 3; \quad \gamma = 6 - i),$$

where A_{ij} are elastic constants; τ_{ij} are stress components corresponding to the sought displacement components u_i; the operators ∂_j are defined by the equalities (2.6).

Substituting the quantities (3.1), (3.2) and the corresponding strain components e_{ij}^0 into the equilibrium equations and the Saint-Venant compatibility conditions, as well as into the boundary conditions indicated at the end of § 1, we obtain that the sought τ_{ij} and u_i must satisfy the equations

$$\sum_{j=1}^{3}\partial_j\tau_{ij} + \xi_3\left[(A_{3i} + A_{\gamma\gamma})\partial_i\partial_1\varphi + i\,\frac{3-i}{2}(A_{36} + A_{45})\partial_\alpha\partial_1\varphi + A_{4\gamma}\right] + \frac{A_{33}}{2}(i-1)(i-2)\partial_1\varphi = 0 \qquad (3.3)$$

in each of the regions t_j $(j = 0, 1, \dots, m)$ and the boundary conditions:

$$[\tau_{in}]_j - [\tau_{in}]_0 = \frac{i}{2}(i-3)\xi_3\left\{\left[(A_{\gamma 5}\partial_1\varphi + A_{4\gamma}\partial_2\varphi - A_{\gamma\gamma}\xi_2 + A_{4\gamma}\xi_1)\cos(n, \xi_1) + A_{36}\partial_1\varphi\cos(n, \xi_\alpha) + A_{3i}\partial_1\varphi\cos(n, \xi_i)\right]_j - [\text{ident}]_0\right\} \qquad (3.4)$$

on the surfaces (2.1) and (2.2) $(j = 1, 2, \dots, m+1;\ [\,]_{m+1} \equiv 0)$;

$$[u_i]_j = [u_i]_0$$

on the surfaces (2.2) $(j = 1, 2, \dots, m)$, where

$$\tau_{in} = \sum_{j=1}^{2} \tau_{ij} \cos(n, \xi_j), \tag{3.5}$$

$i = 1, 2, 3;\ \gamma = 6 - i;\ \alpha = 3 - i$; the symbols $[\,]_j$ and $[\,]_0$ denote the limiting values on the indicated surfaces of the expressions enclosed in square brackets, taken respectively from the regions t_j $(j = 1, 2, \ldots, m)$ and t_0. In addition, the strain components e_{ij} corresponding to the components τ_{ij} will satisfy the Saint-Venant compatibility conditions with respect to the variables ξ_i.

Using the results of § 1, it is easy to see that a particular solution τ_{ij}^1 of equations (3.3) will have the form

$$\tau_{11}^1 = \left[\partial_2^2 \Psi_0 + A_{55}\xi_2 - A_{45}\xi_1 - (A_{31} + A_{55})\partial_1\varphi\right]\xi_3,$$

$$\tau_{22}^1 = \left[\partial_1^2 \Psi_0 - (A_{44} + A_{32})\partial_1\varphi\right]\xi_3,$$

$$\tau_{12}^1 = -\left[\partial_1\partial_2 \Psi_0 + (A_{36} + A_{45})\partial_1\varphi + A_{44}\xi_1\right]\xi_3,$$

$$\tau_{33}^1 = \sigma_1\tau_{11}^1 + \sigma_2\tau_{22}^1 + \sigma_3\tau_{12}^1, \tag{3.6}$$

$$\tau_{i3}^1 = A_{\gamma\gamma}\partial_1\Phi_0 + A_{4\gamma}\partial_2\Phi_0 - \frac{1}{2}\left[\int \left(\sigma_i^*\partial_\alpha\varphi + \sigma_1\partial_2^2\Psi_0 + \sigma_2\partial_1^2\Psi_0 - \sigma_3\partial_1\partial_2\Psi_0\right) d\xi_i \right.$$

$$\left. + \frac{1}{\alpha}\xi_1\xi_i(\sigma_3 A_{44} + \sigma_1 A_{45}) - \frac{\sigma_1}{i}A_{55}\xi_2\xi_i - \frac{1}{A_{\gamma\gamma}}\sigma_\alpha^*\xi_\alpha^2\right] + (-1)^\alpha\partial_\alpha\Phi_1$$

$$(i = 1, 2;\ \gamma = 6 - i;\ \alpha = 3 - i),$$

where

$$\sigma_1^* = 4\beta_{22}A_*A_{45} + A_{44}A_{55}\sigma_3 - 2A_{45}A_{55}\sigma_1 - 2A_{55}\beta_{12}A_* + 2\beta_{23}A_*(A_{45} - A_{44}),$$

$$\sigma_2^* = 4\beta_{11}A_*A_{55} - A_{45}A_{55}\sigma_1 + 2A_{45}A_*(\beta_{33} + \beta_{12}), \tag{3.7}$$

$$\sigma_3^* = E - A_{33} - A_{55}\sigma_1 - A_{44}\sigma_2 - A_{45}\sigma_3,$$

$\Phi_i(\xi_1,\xi_2)$ and $\Psi_0(\xi_1,\xi_2)$ are particular solutions, respectively, of

$$\Delta_1\Phi_0 = -A_{33}\partial_2\varphi, \qquad \Delta_2\Psi_0 = (\beta^*_{11}\partial_1^2 - \beta^*_{13}\partial_1\partial_2 + \beta^*_{12}\partial_2^2)\partial_1\varphi,$$

equations:

$$\Delta_1\Phi_1 = \sum_{i=1}^{2}(-1)^i\partial_\alpha^2\left[(\beta_{1i}A_* - \frac{1}{2}\sigma_3^*A_{kk})\int\varphi\,d\xi_2 -\right.$$

$$\left. -\left(\frac{1}{2}A_{kk}\sigma_i + \beta_{ii}A_*\right)\partial_\alpha\int\Psi_0\,d\xi_i - (A_{kk}\sigma_3 + \beta_{i3})\Psi_0\right] + \frac{1}{2}(A_{44}\sigma_2 - A_{55}\sigma_1)\partial_1\partial_2\Psi_0,$$

$$(k = 3+i; \quad \alpha = 3-i). \tag{3,8}$$

Here the operators Δ_j, defined by the equalities (1,6), are taken with respect to the variables ξ_i; A_* and σ_j^* are defined by the equalities (1,1) and (3,7); by β^*_{1j} the following quantities are denoted:

$$\beta^*_{1j} = A_{55}\beta_{j1} + A_{44}\beta_{j2} + A_{45}\beta_{j3} + \sigma_j \quad (j = 1,2,3). \tag{3,9}$$

The stress components τ_{ij}, satisfying equations (3,1) and the boundary conditions (3,4), shall be represented in the form of the sum

$$\tau_{ij} = \tau^1_{ij} + \tau^*_{ij} \quad (i,j = 1,2,3), \tag{3,10}$$

where τ^1_{ij} are defined by the equalities (3,6), and τ^*_{ij} are to be determined.

Substituting the expressions (3,10) and the corresponding strain components e_{ij} and displacements u_i into the Saint-Venant compatibility conditions, into equations (3,3), and into the boundary conditions (3,4), we obtain that τ^*_{ij} and the strain components e^*_{ij} in each of the regions t_j ($j = 0,1,\dots,m$) of the compound cylindrical body t must satisfy, in the variables ξ_i, the equilibrium equations and the Saint-Venant compatibility conditions, as well as the following boundary conditions:

$$[\tau^*_{in}]_j - [\tau^*_{in}]_0 = \left[\frac{i(i-3)}{2}\xi_3\left(A_{4\gamma}\partial_2\varphi\cos(n,\xi_1) - A_{4\gamma}\partial_1\varphi\cos(n,\xi_2) -\right.\right.$$

$$\left.\left. -A_{4k}\xi_i\cos(n,\xi_\alpha) - \frac{\partial}{\partial s}\partial_\alpha\Psi_0\right) - \frac{1}{2}(i-1)(i-2)\tau^1_{3n}\right]_j - [\text{ident}]_0 \tag{3,11}$$

$$(j = 1,2,\dots,m+1; \quad i = 1,2,3; \quad \gamma = 6-i; \quad k = 3+i,$$

$$\alpha = 3 - i; \quad [\ ]_{m+1} \equiv 0)$$

on the surfaces (2,1) and (2,2),

$$[u_i^*]_j - [u_i^*]_0 = \sum_{k=0}^{1} \xi_3^k \omega_{ik}(\xi_1, \xi_2) \quad (j = 1, 2, \dots, m; \quad i = 1, 2, 3)$$

on the surfaces (2,2), where the operator τ_{in} is defined by equality (3,5); $\partial/\partial s$ is the operation of differentiation with respect to the arc of the corresponding curve; τ_{ij}^1 are defined by the equalities (3,6); φ is the torsion function of the compound anisotropic cylindrical body t; Ψ_j are particular solutions of equation (3,8), and the values ω_{ik} are easy to construct if, from (3,6), one restores the displacements u_i^1.

Thus, the determination of the unknowns τ_{ij}^* has been reduced to the Almansi problem for the compound anisotropic cylindrical body t with a lateral load varying according to a polynomial law with respect to the axial coordinate ξ_3. This problem is solved in paper (5), in which explicit expressions are given for the components τ_{ij}^*.

The stress components (3,2), in which τ_{ij} are defined by the equalities (3,10), on the end $x_3 = l$ of the compound anisotropic bent beam t' will generally not be equivalent to a twisting couple with moment L. Therefore, as in the case of an isotropic body ([1]), the components (3,10) must be supplemented by the corresponding solutions of the Saint-Venant problems for the compound anisotropic cylindrical body t ([4]).

Tbilisi Mathematical Institute
named after A. M. Razmadze
Academy of Sciences of the Georgian SSR

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Note: Figure translations are in progress. See original paper for figures.