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Abstract

Full Text

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ON SOME CAUCHY PROBLEMS WITH INITIAL DATA ON THE LINE OF DEGENERATION

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MATHEMATICS

1. Consider the equations

$$LU \equiv y^\alpha U_{xx} - U_{yy} + a(x,y)U_x + b(x,y)U_y + c(x,y)U = f(x,y) \qquad (1)$$

$$(y > 0,\ \alpha > 0),$$

$$LLU = f(x,y) \qquad (y > 0,\ \alpha \geq 2) \qquad (2)$$

with initial data

$$\lim_{y\to 0} \frac{\partial^i U}{\partial y^i} = \tau_i(x) \quad (x \in AB;\ i = 0,1 \text{ for } (1);\ i = 0,1,2,3 \text{ for } (2)). \qquad (3)$$

Let D be the domain bounded by a segment of the x-axis, by the characteristics AC and BC of equation (1), and by the straight line $y = \delta$, where $\delta > 0$ is sufficiently small.

Denote by C_n^r the set of functions $\psi(x,y)$ for which $y^{i-n}\partial^j\psi(x,y)/\partial x^{j-i}\partial y^i$ $(0 \leq i \leq j \leq r)$ are continuous in $\bar{D}$.

Theorem 1. Equation (1) with initial data (3) has a unique solution in C_0 if: 1) $a \in C^{2N+r}_{\alpha/2-1}$, $b \in C^{2N+r}_{\sigma_0}$, $c \in C^{2N+r}_{\sigma_1}$, $f \in C^{2N+r*}_{\sigma_2}$, where $\sigma_i > -1$ $(i = 0,1,2)$,

$$N = 1 + \left[\frac{\max\{2r-2;\ M + \alpha - 3 - 2\min(\alpha/2-1, \sigma_0, \sigma_1, \sigma_2) + \sqrt{(M-1)^2 + \alpha(\alpha+2)}\}}{2(2+\min(\alpha/2-1, \sigma_1))}\right]^{**},$$

$$M = \sup_{AB}\left|\lim_{y\to 0} y^{1-\alpha/2} a\right|, 2)\, \tau_i(x) \qquad (i = 0,1),$$

together with their derivatives up to order $2N + r + 2$ inclusive, are continuous on AB.

For $0 < \alpha < 2$ the solution of problem (1), (3) was obtained by I. S. Berezin ([1]), and for $\alpha = 2$ and $|a(x,y)| < M$ by the author ([6]). Under other restrictions on the coefficients and on the sought solutions, problem (1), (3) was considered by M. H. Protter ([4]), G. Hellwig ([5]), and others.

In the proof of Theorem 1 some ideas from the works ([3,6]) are used. A new unknown function is introduced,

$$Y = U - \tau_0(x) - y\tau_1(x) - \sum_{k=1}^{N} \varphi_k(x,y),$$

where

$$\varphi_k = -A^{-1}R\varphi_{k-1} \equiv \int_0^y d\tau \int_0^\tau \exp\left[\int_t^\tau b(x,\xi)\,d\xi\right] R\varphi_{k-1}(x,t)\,dt, \qquad R\varphi =$$

$$= y^\alpha \varphi_{xx} + a\varphi_x + c\varphi,$$

and by $R\varphi_0(x,y)$ is denoted the function

$$L[\tau_0(x) + y\tau_1(x)] - f(x,y).$$

* These conditions can be weakened.

** $[x]$ denotes the integer part of the number x.

Problem (1), (3) passes into

$$LV = -R\varphi_N; \tag{4}$$

$$\lim_{y\to 0} V = \lim_{y\to 0} V_y = 0, \tag{5}$$

where $R\varphi_N = O(y^{p-2})$ as $y \to 0$, $p = 2 + \min(\alpha/2 - 1, \sigma_0, \sigma_1, \sigma_2) + N(2 + \min(\alpha/2 - 1, \sigma_1))$. Condition (5) and equation (4), represented in the form $V = -A^{-1}(RV + R\varphi_N)$, allow us to conclude that $V \in C_p^r$.

From (4), by integrating over the domain bounded by a segment of the x-axis and by characteristics passing through the point $P(x,y) \in \bar{D}$, one can obtain the integral equation $V = TV$, where

$$TV = \frac{1}{2}\Bigg\{ \int_0^y \Big[t^{-\alpha/2}a(s,t) + b(s,t) - \frac{\alpha}{2t}\Big] V(s,t) - \Big(t^{-\alpha/2}a(q,t) - b(q,t) + \frac{\alpha}{2t}\Big) V(q,t)\,dt + \int_0^y t^{-2-\alpha/2}\,dt \int_q^s k(\tau,t)V(\tau,t)\,d\tau + \int_0^y t^{-\alpha/2}\,dt \int_q^s R\varphi_N(\tau,t)\,d\tau \Bigg\},$$

$$s = x - \frac{1}{m}(t^m - y^m), \qquad q = x + \frac{1}{m}(t^m - y^m) \qquad \Big(m = \frac{\alpha+2}{2}\Big),$$

$$k(x,y) = -y^2 a_x + \frac{\alpha}{2} yb - y^2 b_y + y^2 c - \frac{\alpha(\alpha+2)}{4}.$$

If on the set C_p^r we introduce a metric by the formula

$$\rho(V, \widetilde{V}) = \sum_{j=0}^{r} \sum_{i=0}^{j} \delta^j \sup_{\bar{D}} \left| y^{i-p} \frac{\partial^j (V - \widetilde{V})}{\partial x^{j-i} \partial y^i} \right|,$$

then we shall have a complete metric space. By direct verification one can be convinced that the operator T, acting in this space, is a contraction operator.

The uniqueness of the solution of problem (1), (3) follows from the fact that the homogeneous equation corresponding to (1), with zero initial data, has only the zero solution.

Theorem 2. *Equation (2) with initial data (3) has in C_0^r a unique solution if: 1) $a \in C_{\alpha/2-1}^{4N+1}$, $b, c, f \in C_0^{4N+r}$, $a_y, b_y, c_y \in C_0^0$; 2) $\tau_i(x)$ and their derivatives up to order $4N + r + 3$ inclusive are continuous on AB.*

Problem (2), (3) was posed by A. V. Bitsadze ([2]). A special case ($\alpha = 2$) of this problem was considered by me earlier ([7]).

The proof of Theorem 2 is obtained by successively solving two problems:

a) $Lw = f$, $\lim_{y\to 0} w = T_0(x)$, $\lim_{y\to 0} w_y = T_1(x)$, where $T_0(x) = \lim_{x\to 0} LU$, $T_1(x) = \lim_{y\to 0} \partial LU/\partial y$ are determined completely by the coefficients of (2) and the initial data (3);

b) $LU = w$, $\lim_{y\to 0} U = \tau_0(x)$, $\lim_{y\to 0} U_y = \tau_1(x)$.

Problem a), according to Theorem 1, has a unique solution $w \in C_0^{2N+r}$. This w allows one to find in C_0^r the unique solution of problem b), which at the same time is also the unique desired solution of problem (2), (3).

2. Correctness

For the set of functions belonging to C_0^r, introduce a metric in the following way:

$$\rho_{C_0^r}(U,\widetilde{U}) = \sum_{j=0}^{r}\sum_{i=0}^{j}\sup_D\left|y^i\frac{\partial^j(U-\widetilde{U})}{\partial x^{j-i}\partial y^i}\right|.$$

For the set of functions belonging to C^n, by $\rho_{C^n}(\tau_i,\tilde{\tau}_i)$, as usual, we shall understand

$$\sum_{j=0}^{n}\sum_{i=0}^{3}\sup\left|\tau_i^{(j)}(x)-\tilde{\tau}_i^{(j)}(x)\right|.$$

Definition. We shall say that the Cauchy problem is posed correctly of order (r,n) if, for every $\varepsilon>0$, there exists an $\eta>0$ such that $\rho_{C_0^r}(U,\widetilde{U})<\varepsilon$ whenever $\rho_{C^n}(\tau_i,\tilde{\tau}_i)<\eta$.

From the structure of the operators A^{-1} and R it follows

Lemma. The function $R\varphi_N(x,y)$ is representable in the form

$$R\varphi_N(x,y) = f_N(x,y) + \sum_{k=0}^{2N+2}\sum_{i=0}^{1}\alpha_{ik}(x,y)\tau_i^{(k)}(x),$$

where $\alpha_{ik}(x,y)=O(y^{p-2})$ as $y\to 0$, and $f_N(x,y)$ does not depend on $\tau_i(x)$.

Theorem 3. The problem (1), (3), under the conditions of Theorem 1, is posed correctly of order $(r,2N+r+2)$.

For $\delta\le 1$, taking the lemma into account, we shall have

$$\begin{aligned}\rho_{C_0^r}(U,\widetilde{U}) \le \sum_{j=0}^{r}\sum_{i=0}^{j}\sup_D\Bigg|y^i\frac{\partial^j}{\partial x^{j-i}\partial y^i}\Big\{&\tau_0(x)-\tilde{\tau}_0(x)+y\left[\tau_1(x)-\tilde{\tau}_1(x)\right]\\ &+\sum_{k=1}^{N}\left[\varphi_k(x,y)-\widetilde{\varphi}_k(x,y)\right]\Big\}\Bigg| + \frac{1}{\delta^r}\rho(V,\widetilde{V})\\ \le M_0\rho_{C^{2N+r+2}}(\tau_i(x),\tilde{\tau}_i(x)).&\end{aligned}$$

Hence the validity of Theorem 3 follows.

Theorem 4. The problem (2), (3), under the conditions of Theorem 2, is posed correctly of order $(r,4N+r+3)$.

The proof of this theorem is obtained by estimating $\rho_{C_0^r}(U, \widetilde{U})$, taking into account the lemma given above and Theorem 3.

In conclusion we note that, by analogy with the study of the problem (2), (3), one considers the problem

$$L^{nU} = f(x,y), \qquad \lim \partial^i U/\partial y^i = \tau_i(x) \quad (x \in AB,\ i = 0, 1, \dots, 2m-1),$$

where n is any positive integer, and L^{nU} denotes the n-fold successive application of the operator L to the function U.

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Note: Figure translations are in progress. See original paper for figures.