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**Abstract**

**Full Text**

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## THE BEHAVIOR OF METRIZABILITY UNDER QUOTIENT MAPPINGS

*(Presented by Academician P. S. Aleksandrov, 16 I 1965)*

Any space is trivially realized as the image of a metric space. On the other hand, consideration of the simplest decompositions of metric spaces, in which only one element is nondegenerate, leads, generally speaking, to nonmetrizable spaces. This cannot happen if the nondegenerate element is compact, but already when a decomposition contains a countable set of nondegenerate elements, even if they are all compact, the corresponding quotient space may be nonmetrizable. Here we shall consider a broad class of quotient mappings, including, in particular, all quotient compact mappings. The main result of the paper—Theorem 1—establishes when the mappings under consideration carry metrizability into images.

**Definition 1.** A mapping  $f : X \rightarrow Y$  of a metric space  $X$  is called a  $\pi$ -mapping if, for every point  $y \in Y$  and every neighborhood  $O_y$  of it,

$$\rho(X \setminus f^{-1}O_y, f^{-1}y) > 0$$

\*

**Theorem 1.** *A paracompact  $p$ -space (see (1)) that is a quotient  $\pi$ -image of a metric space is metrizable. This result is definitive in the following sense: 1) there exists a weakly paracompact  $p$ -space that is an open bicomact (and, a fortiori, quotient  $\pi$ -) image of a metric space and is not metrizable; 2) there exists a finally compact (and, a fortiori, paracompact) space that is not metrizable and is a quotient finite-to-one (and, a fortiori,  $\pi$ -) image of the line.\*\**

**Proof of Theorem 1.**

**Proposition 1.** *A compact Hausdorff space  $Y$  that is a quotient  $\pi$ -image of a metric space  $X$  is metrizable.*

**Proof.** Let  $f : X \rightarrow Y$  be a quotient  $\pi$ -mapping. Put

$$d(y_1, y_2) = \inf_{x_1 \in f^{-1}y_1, x_2 \in f^{-1}y_2} \rho(x_1, x_2).$$

Since  $Y$  is a Hausdorff space and  $f$  is a  $\pi$ -mapping,  $d$  satisfies the conditions: 1)  $d(y_1, y_2) = 0 \leftrightarrow y_1 = y_2$ ; 2)  $d(y_1, y_2) = d(y_2, y_1)$ ; 3) if  $y \in V$  and  $V$  is open in

$Y$ , then there is an  $\varepsilon > 0$  such that  $O_\varepsilon(y) \subseteq V^{***}$ ; 4) if  $P$  is a nonclosed subset of  $Y$ , then there is a  $y \in [P] \setminus P$  for which  $d(y, P) = 0$ .

**Lemma 1.** Put  $F_n(y) = [O_{1/n}(y)]$ ; then, for a fixed arbitrary  $y \in Y$  and for every  $\varepsilon > 0$ , there is a number  $N(\varepsilon)$  such that  $F_n(y) \subseteq O_\varepsilon(y)$  for all  $n > N(\varepsilon)$ .

**Proof of Lemma 1.** Suppose the contrary, and let  $\varepsilon > 0$  and  $y_n \in F_n(y)$  be chosen so that  $d(y, y_n) \geq \varepsilon$ ,  $n = 1, 2, \dots$ . We shall show,

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\* This condition is due to V. Ponomarev (5). He used the term weakly uniformly continuous mapping.

\*\* In this note all spaces are assumed in advance to be completely regular, and all mappings continuous.

\*\*\* By  $O_\varepsilon(y)$  is denoted the set of all  $y' \in Y$  such that  $d(y, y') < \varepsilon$ .

that the sequence  $\{y_n\}$  converges to  $y$ . Indeed, let  $V$  be an arbitrary neighborhood of the point  $y$ . Then, by the regularity of  $Y$ , there is a set  $U$ , open in  $Y$ , such that  $y \in U \subset [U] \subset V$ . Applying 3) to  $y$  and  $U$ , we find an  $N$  such that  $O_{1/N}(y) \subset U$ . Then  $F_N(y) = [O_{1/N}(y)] \subset [U] \subset V$ . From the obvious relation  $F_{n_1}(y) \subset F_{n_2}(y)$  for  $n_1 > n_2$  it follows that  $y_n \in V$  for all  $n > N$ . Thus  $y$  is the unique limiting point of the set  $P = \{y_n\}$ ; clearly  $y \notin P$ . By 4), one must have  $d(y, P) = 0$ —a contradiction to  $d(y, P) \geq \varepsilon > 0$ . Lemma 1 is proved.

For fixed  $\varepsilon > 0$ , choose for each point  $y \in Y$  a number  $n(\varepsilon, y)$  such that

$$F_{n(\varepsilon, y)}(y) \subset O_\varepsilon(y).$$

Define a sequence  $\{y_\alpha\}$  of points of the space  $Y$ , numbered by ordinal numbers, as follows. Put  $y_0 = y$  arbitrarily. Suppose that, for all  $\beta < \alpha_0$ , the points  $y_\beta$  have been defined. Put

$$Q_{\alpha_0} = \bigcup_{\beta < \alpha_0} F_{n(\varepsilon, y_\beta)}(y_\beta).$$

Two cases are possible: 1)  $Q_{\alpha_0} = Y$ ; then we shall regard the process of constructing the sequence  $\{y_\alpha\}$  as completed; 2)  $Y \setminus Q_{\alpha_0} \neq \Lambda$ ; then choose as  $y_{\alpha_0}$  some point of  $Y \setminus Q_{\alpha_0}$ . Sooner or later this process will terminate (since  $y_{\alpha_1} \neq y_{\alpha_2}$  when  $\alpha_1 \neq \alpha_2$ ); suppose that by the time this happens the sequence contains an uncountable set of points, and derive a contradiction from this supposition. Indeed, to each point  $y_\alpha \in \{y_\alpha\}$  there corresponds a number  $n_\alpha = n(\varepsilon, y_\alpha)$ . If the set  $\{y_\alpha\}$  is uncountable, then for some integer  $k$  the set  $P_k$  of those  $y_\alpha \in \{y_\alpha\}$  for which  $n_\alpha = k$  is uncountable. We shall show that the distance between any two points  $y', y'' \in P_k$  is not less than  $1/k$ .

Indeed,  $y' = y_{\alpha_1} \in \{y_\alpha\}$ ,  $y'' = y_{\alpha_2} \in \{y_\alpha\}$  for some values  $\alpha_1$  and  $\alpha_2$  of the index  $\alpha$ . Suppose, for definiteness, that  $\alpha_1 < \alpha_2$ . Then

$$F_{n_{\alpha_1}}(y_{\alpha_1}) = F_k(y_{\alpha_1}) \not\ni y_{\alpha_2},$$

by the construction of the sequence  $\{y_\alpha\}$ ; hence  $O_{1/k}(y_{\alpha_1}) \not\supseteq y_{\alpha_2}$ , i.e.

$$d(y_{\alpha_1}, y_{\alpha_2}) = d(y', y'') > 1/k.$$

We shall show that  $P_k$ , and any of its subsets  $P'_k$ , are closed in  $Y$ . For this, consider

$$T'_k = f^{-1}P'_k; \quad T'_k = \bigcup_{y_\alpha \in P'_k} f^{-1}y,$$

where

$$\rho(f^{-1}y', f^{-1}y'') \geq \varepsilon, \quad y', y'' \in P'_k.$$

Therefore  $\{f^{-1}y \mid y \in P'_k\}$  is a discrete system of closed subsets in  $X$ , and hence  $T'_k$  is closed in  $X$ . By the quotientness of  $f$ ,  $P'_k$  is also closed in  $Y$ . We have obtained a contradiction with the compactness of  $Y$  and the infinitude of the set  $P_k$ . The contradiction obtained means that the sequence  $\{y_\alpha\}$  terminates while still countable. Then

$$\gamma_\varepsilon = \{F_{n(\varepsilon, y)}(y) \mid y \in \{y_\alpha\}\}$$

is a countable covering of the space  $Y$  by closed sets of radius  $\leq \varepsilon$ . Choose, for each

$$\varepsilon_k = 1/k, \quad k = 1, 2, \dots,$$

some countable covering  $\gamma_{\varepsilon_k}$  of the space  $Y$  by closed sets of radius  $< 1/k$ .

Put

$$S = \bigcup_{k=1}^{\infty} \gamma_{\varepsilon_k};$$

denote by  $\tilde{S}$  the totality of all possible finite intersections of elements of  $S$ . The system  $\tilde{S}$  is countable; we shall show that it forms a net in  $Y$ .

Indeed, let  $y \in U$ , where  $U$  is open in  $Y$ . Choose from each  $\gamma_\varepsilon$  such an  $s_k$  that  $y \in s_k$ , and put

$$s'_j = \bigcap_{i=1}^j s_i.$$

We shall show that at least one element of the resulting system  $\{s'_k\}$  is contained in  $U$ . Suppose the contrary. Then

$$\bigcap_{k=1}^{\infty} s'_k \cap (Y \setminus U) \neq \Lambda;$$

let

$$y_0 \in \bigcap_{k=1}^{\infty} s'_k \cap (Y \setminus U)$$

and let  $y_n$ ,  $n = 1, 2, \dots$ , be some centers of the elements  $s'_n$ .<sup>\*</sup> Then

$$\lim_{n \rightarrow \infty} d(y_n, y_0) = 0$$

and

$$\lim_{n \rightarrow \infty} d(y_n, y) = 0.$$

This is impossible—

\* This means that  $O_{1/n}(y_n) \supset s'_n$ . Such points  $y_n$  exist by the definition of the system  $\gamma_\varepsilon$ .

but, by virtue of the Hausdorffness of  $Y$ , the relation  $y_0 \neq y$ , and condition 3. Hence  $S$  is a net in  $Y$ . But a compact Hausdorff space with a countable net has a countable base <sup>(2)</sup>. Proposition 1 is proved.

Now, in the  $p$ -space  $Y$ , every point  $y$  is contained in some bicomcompactum of countable character (see <sup>(1)</sup>). If  $Y$  is a quotient  $\pi$ -image of a metric space, then, by Proposition 1, this bicomcompactum is a compactum, whence it follows that the first axiom of countability is fulfilled in  $Y$  <sup>(1)</sup>. From this it is easy to derive that the function  $d(y_1, y_2)$ ,  $y_1, y_2 \in Y$ , defined as the distance between the preimages of  $y_1$  and  $y_2$ , agrees with the topology of  $Y$  in the following sense:  $y \in [M]$  if and only if  $d(y, M) = 0$ .

Therefore Theorem 1 will be proved if we establish

**Proposition 2.** *Let  $X$  be a paracompact feathered space with a symmetric  $\rho$ . Then  $X$  is metrizable.*

**Proof.** Consider any  $bX \supset X$ , a bicomcompact extension of the space  $X$ , and let  $\varphi = \{\gamma_n\}$ ,  $\gamma_n = \{U_\alpha^n, \alpha \in M_n\}$ , be a feathering of  $X$  in  $bX$  (see <sup>(1)</sup>). Since the intersection of a finite number of open covers of a space is again an open cover of the space, and the intersection of a finite number of locally finite covers is a locally finite cover, there exists a sequence  $\psi = \{\lambda_n\}$  of locally finite covers  $\lambda_n = \{V_\alpha^n, \alpha \in L_n\}$  of the space  $X$ , satisfying the conditions: a) for each  $V_\alpha^n \in \lambda_n$  there is a point  $x \in X$  such that  $[V_\alpha^n]_X \subset O_{1/n}x$ ; b) the closures in  $bX$  of the elements of  $\lambda_n$  are contained in elements of  $\gamma_n$ ; c)  $\lambda_n \cap \lambda_{n+1} = \lambda_{n+1}$ ,  $n = 1, 2, \dots$

Condition c) is needed only in order to ensure the following property of the sequence  $\{\lambda_n\}$ : c') for any point  $x \in X$  and any element  $V_{\alpha_n}^n \ni x$ ,  $V_{\alpha_n}^n \in \lambda_n$ , there is a  $V_{\alpha_{n+1}}^{n+1} \in \lambda_{n+1}$  for which

$$x \in V_{\alpha_{n+1}}^{n+1} \subset V_{\alpha_n}^n.$$

We shall prove that

$$B = \bigcup_{n=1}^{\infty} \lambda_n$$

is a base of the space  $X$ . Let a point  $x \in X$  and its neighborhood  $Ox$  be arbitrary. By c'), there is a sequence  $\{V_{\alpha_n}^n, \alpha_n \in L_n, n = 1, 2, \dots\}$  such that  $V_{\alpha_n}^n \subset V_{\alpha_{n-1}}^{n-1}$  and

$$x \in \bigcap_{n=1}^{\infty} V_{\alpha_n}^n.$$

Denote by  $G$  some set open in  $bX$  for which  $G \cap bX = Ox$ .

Suppose that for every  $n$ ,

$$V_{\alpha_n}^n \cap (X \setminus G) \neq \Lambda.$$

Then, a fortiori,

$$F_n = [V_{\alpha_n}^n]_{bX} \cap (bX \setminus G) \neq \Lambda.$$

The sequence of bicompecta  $\{F_n\}$  is decreasing; hence

$$\Phi = \bigcap_{n=1}^{\infty} F_n \neq \Lambda.$$

But since  $x \in [V_{\alpha_n}^n]_{bX}$ , it follows from b) that

$$[V_{\alpha_n}^n]_{bX} \subset \gamma_n(x).$$

Since  $\varphi = \{\gamma_n\}$  is a feathering of  $X$  in  $bX$  (see (1)), we have

$$\bigcap_{n=1}^{\infty} \gamma_n(x) \subset X.$$

Then also

$$\Phi \subset \bigcap_{n=1}^{\infty} [V_{\alpha_n}^n]_{bX} \subset X.$$

At the same time,

$$\Phi \subset X \setminus G = X \setminus Ox.$$

By a) there exists a sequence  $\{x_n\}$ ,  $x_n \in X$ , for which  $O_{1/n}x_n \supset [V_{\alpha_n}^n]_X$ . Let now  $x'$  be any point of  $\Phi$ . Then  $x' \neq x$ ,  $x \in [V_{\alpha_n}^n]_X$ , and

$$x' \in [V_{\alpha_n}^n]_{bX} \cap X = [V_{\alpha_n}^n]_X.$$

Consequently,

$$\rho(x_n, x) < 1/n \quad \text{and} \quad \rho(x_n, x') < 1/n, \quad n = 1, 2, \dots$$

However,  $X$  is a Hausdorff space; therefore the points  $x$  and  $x'$  possess disjoint neighborhoods  $O_1x$  and  $O_1x'$ . By the definition of the topology in  $X$  (it is generated by the symmetric given on  $X$ ), for some sufficiently large number  $N$  we shall have

$$O_{1/N}(x) \subset O_1x \quad \text{and} \quad O_{1/N}(x') \subset O_1x'.$$

Then

$$O_{1/N}(x) \cap O_{1/N}(x') = \Lambda,$$

a contradiction to the fact that  $\rho(x, x_N) < 1/N$  and  $\rho(x', x_N) < 1/N$ , and, consequently,

$$x_N \in O_{1/N}(x) \cap O_{1/N}(x').$$

This proves that, for some number  $n_0$ ,

$$V_{\alpha_{n_0}}^{n_0} \cap (X \setminus G) = \Lambda,$$

whence, in view of the fact that  $G \cap X = Ox$  and  $x \in V^{n_0} \subset X$ , we obtain  $x \in V_{\alpha_{n_0}}^{n_0} \subset Ox$ . Thus it has been proved that  $B = \bigcup_{n=1}^{\infty} \lambda_n$  is a base of the space  $X$ . But all  $\lambda_n$  are locally finite. Hence, by the Nagata–Smirnov criterion (see (6)), the space  $X$  is metrizable. Proposition 2a has been proved, and hence Theorem 1 has also been proved.

The technique developed for the proof of Theorem 1 also makes it possible to formulate several special results on spaces with a symmetric. We give several definitions. A function  $d(y', y'')$ , defined on the set of pairs of points of a space  $Y$ , will be called a weak symmetric if conditions 1–4 from the proof of Proposition 1 are satisfied. We shall say that a weak symmetric  $d$  satisfies the weak Cauchy condition if in every nonclosed set, for each  $\varepsilon > 0$ , there is such a pair of points  $y', y''$  that  $d(y', y'') < \varepsilon$ . As we have seen, such weak symmetrices arise under quotient  $p$ -mappings of metric spaces. We shall call a sequence  $\{y_n\}$  of points of a space  $Y$  with a weak symmetric  $d$  fundamental if for every  $\varepsilon > 0$  there are a point  $y \in Y$  and a number  $N$  such that  $d(y, y_n) < \varepsilon$  for  $n > N$ . A space with a weak symmetric is called complete in the Cauchy sense if every fundamental sequence in it converges\*.

In essence, we have proved above

**Theorem 2.** \*A paracompact  $p$ -space with a weak symmetrix satisfying the weak Cauchy condition is metrizable\*\*.\*

**Problem.** Is an arbitrary bicompactum with a weak symmetrix (see (4)) metrizable? Let us note that for finally compact spaces the answer is negative.

It is known that there exist nonmetrizable paracompacta with a symmetrix\*\*\*.

**Theorem 3.** *Every paracompactum lying in a space with a complete symmetrix is metrizable. Theorem 3 is reduced, with the aid of the theorem on approximation of paracompact spaces lying in feathered spaces (see (1)), to Proposition 2 after the lemma: every space with a complete symmetrix is complete in the sense of Čech.*

**Corollary.** *There exists a paracompactum with a symmetrix which cannot be embedded in any space with a complete symmetrix.*

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## CITED LITERATURE

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\* This definition is equivalent to one of McAuley's definitions (3).

\*\* The center of gravity in the proof of this theorem is for bicompacta. Theorem 2 is more general than Theorem 1.

\*\*\* The fact that every paracompactum with a complete symmetrix is metrizable was proved earlier by J. Ceder (7).

*Note: Figure translations are in progress. See original paper for figures.*

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