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Abstract

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FUNCTIONALLY INVARIANT SOLUTIONS OF SECOND-ORDER EQUATIONS

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The question of functionally invariant solutions (f.i. solutions) of a second-order equation was considered in ($^{1-3}$) in connection with isotropic congruences of lines in three-dimensional, generally speaking Riemannian, and also in n-dimensional Euclidean space. There it was established that to every isotropic congruence there corresponds a second-order equation with principal part in the form of the Laplace operator (in a Riemannian metric), possessing f.i. solutions, and a condition for the isotropy of the congruence was obtained.

In the present article, proceeding by another route, we shall find conditions, necessary and sufficient, for a given elliptic equation with three independent variables to possess f.i. solutions, and we shall indicate a method for constructing such solutions. The entire discussion will be carried out in Cartesian coordinates x_1, x_2, x_3, without using the above-mentioned connection between f.i. solutions and congruences of lines.

Let us have the elliptic equation

$$\operatorname{div} A \operatorname{grad} w + (Z, \operatorname{grad} w) = 0, \tag{1}$$

where the elements of the symmetric, positive (as an operator) matrix A and the components of the vector Z are sufficiently smooth real functions. The determinant $|A|$ may, without loss of generality, be assumed equal to unity; this simplifies some formulas. The scalar product is introduced in the usual way: $(p, q) = p_k q_k$ (including for complex vectors).

We establish necessary and sufficient conditions for the existence of f.i. solutions of (1), regarding the matrix A as given.

Suppose that (1) has an f.i. solution $w = u + iv \neq \text{const}$, i.e., such a solution that any analytic function $F(w)$ also satisfies (1). Then it is necessary that

$$(A \operatorname{grad} w, \operatorname{grad} w) = 0. \tag{2}$$

Introduce the real symmetric matrix $B = \sqrt{A}$. According to what was said above, we may assume $|B| = 1$. From (2) we obtain:

$$(B\operatorname{grad} w, B\operatorname{grad} w) = 0. \tag{3}$$

Every vector q, orthogonal to itself, in three-dimensional space can be reduced to the form $q = \omega\vec{\delta}(1, \operatorname{sh}\lambda, -i\operatorname{ch}\lambda)$, where ω is some scalar (complex) function, and the components of the vector $\vec{\delta}$ along the axes of the Cartesian coordinate system (written in parentheses next to the vector) depend on one complex function λ. Thus:

$$B\operatorname{grad} w = \omega\vec{\delta}(1, \operatorname{sh}\lambda, -i\operatorname{ch}\lambda). \tag{4}$$

From (4) we find

$$(\mathfrak{B}\vec{\delta}, \operatorname{rot}\mathfrak{B}\vec{\delta}) = 0, \tag{5}$$

where $\mathfrak{B} = B^{-1}$, and also the three differential equations satisfied by w:

$$(B\vec{\delta}, \nabla)w = 0, \qquad (B\vec{\gamma}, \nabla)w = 0, \tag{6}$$

$$Lw \equiv \operatorname{div} A \operatorname{grad} w = \frac{1}{(\vec{\delta}, \vec{\delta}^*)}\{(B\vec{\gamma}, \nabla)\lambda + i(\vec{\delta}, T\vec{\gamma})\}(B\vec{\delta}^*, \nabla)w. \tag{7}$$

Here $\vec{\gamma}(0, \operatorname{ch}\lambda, -i\operatorname{sh}\lambda) = \vec{\delta}_\lambda$, and the asterisk denotes the complex conjugate quantity. The vectors $\vec{\delta}$, $\vec{\gamma}$, $\vec{\delta}^*$ form a complete system of vectors in three-dimensional space. The matrix T has components $\{T\}_{lk} = (\mathfrak{B}_l, \operatorname{rot}\mathfrak{B}_k)$, where $\mathfrak{B}_l$ is the vector whose components are the elements of the l-th row of the matrix $\mathfrak{B}$. The matrix T, generally speaking, is nonsymmetric.

By virtue of (6), (7), the general form of the second-order equation with prescribed principal part Lw satisfied by w will be

$$Lw + \left(\alpha B\vec{\delta} + \beta B\vec{\gamma} - \frac{1}{(\vec{\delta}, \vec{\delta}^*)}\{(B\vec{\gamma}, \nabla)\lambda + i(\vec{\delta}, T\vec{\gamma})\}B\vec{\delta}^*, \nabla\right) w = 0, \tag{8}$$

where α, β are arbitrary functions.

Relation (5) is the condition of quasipotentiality of the vector $\mathfrak{B}\vec{\delta}$; at the same time it is the condition of completeness of the system (6). After transformation, (5) takes the form

$$(B\vec{\delta}, \nabla)\lambda = -i(\vec{\delta}, T\vec{\delta}). \tag{9}$$

Since (1) possesses f.i. solutions, the vector Z, in view of (8), must have the form

$$Z = \alpha B\vec{\delta} + \beta B\vec{\gamma} - \frac{1}{(\vec{\delta},\vec{\delta}^{*})}\{(B\vec{\gamma},\nabla)\lambda + i(\vec{\delta},T\vec{\gamma})\}B\vec{\delta}^{*}, \tag{10}$$

where α, β are certain functions of the coordinates. Since Z is a real vector, it must be that

$$\alpha = -\frac{1}{(\vec{\delta},\vec{\delta}^{*})}\{(B\vec{\gamma}^{*},\nabla)\lambda^{*} - i(\vec{\delta}^{*},T\vec{\gamma}^{*})\} - \beta\ \mathrm{th}(\mathrm{Re}\,\lambda),$$

where β is a real function. From (10) we have

$$(B\vec{\gamma},\nabla)\lambda = -i(\vec{\delta},T\vec{\gamma}) - (\mathfrak{B}Z,\vec{\delta}). \tag{11}$$

Computing, with the aid of (9), (11), the commutator $(B\vec{\delta},\nabla)(B\vec{\gamma},\nabla)\lambda - (B\vec{\gamma},\nabla)(B\vec{\delta},\nabla)\lambda$, while taking into account the completeness of the system (6), we find

$$(\vec{\delta},R\vec{\delta}) = 0. \tag{12}$$

Here the symmetric matrix R does not contain λ. The components of the vector Z and their first derivatives enter into R, as do the elements of the matrix B and their derivatives up to and including second order. We shall not give here the rather cumbersome expressions for the elements of the matrix R.

Applying the operators $(B\vec{\delta},\nabla)$, $(B\vec{\gamma},\nabla)$ to (12) and using (9), (11), we obtain a system of three equations into which the derivatives of λ do not enter:

$$(\vec{\delta},R\vec{\delta}) = 0, \qquad (\vec{\delta},[(B\vec{\delta},\nabla)R + 2iRMT]\vec{\delta}) = 0,$$

$$(\vec{\delta},[(B\vec{\gamma},\nabla)R]\vec{\delta} + 2iRMT\vec{\gamma} + 2RM\mathfrak{B}Z) = 0. \tag{13}$$

Here $\{(\vec{B}\vec{\delta},\nabla)R\}_{kl} = (\vec{B}\vec{\delta},\nabla)\{R\}_{kl}$, and the matrix M has the form

$$M = \begin{Vmatrix} 0 & \mathrm{ch}\,\lambda & -i\,\mathrm{sh}\,\lambda \\ -\mathrm{ch}\,\lambda & 0 & i \\ i\,\mathrm{sh}\,\lambda & -i & 0 \end{Vmatrix},$$

so that $M\vec{\gamma} = \vec{\delta}, \quad M\vec{\delta} = (0,0,0)$.

Eliminating λ from (13), we arrive at two complex (or four real) relations connecting the elements of the matrix B and the components of the vector Z. When these relations are satisfied, λ is uniquely determined. We shall not give here the results of eliminating λ and the formula for λ, in view of their unwieldiness.

The fulfillment of the mentioned relations between the elements of the matrix B and the components of the vector Z is not only a necessary but also a sufficient condition for the existence of f.i. solutions of equation (1). When these relations are satisfied, a function $\lambda(x)$ satisfying (9) is found, which ensures the completeness of system (6). Finding the f.i. solution from (6) reduces to the successive solution of two ordinary equations. For this purpose we replace system (6) by the equivalent Jacobi system

$$\partial w/\partial x_1 - q_1 \partial w/\partial x_3 = 0, \qquad \partial w/\partial x_2 - q_2 \partial w/\partial x_3 = 0, \tag{14}$$

where $q_k = (\vec{\mathfrak{B}}_k, \vec{\delta})/(\vec{\mathfrak{B}}_3, \vec{\delta})$. The first of equations (14) gives $w = S(x_2, \Phi)$, where S is an arbitrary function, and Φ is found as an integral of the ordinary equation $dx_3/dx_1 = -q_1$. With the aid of the second equation (14), the f.i. solution is found as an integral of the ordinary equation $d\Phi/dx_2 = Q(x_2, \Phi)$, where $Q = \partial\Phi/\partial x_2 - q_2 \partial\Phi/\partial x_3$. The completeness condition for system (6) (and hence also (14)) guarantees that the right-hand side in the last equality indeed depends only on x_2, Φ.

Remark 1. The function λ and the vectors $\vec{\delta}, \vec{\gamma}$ have a simple geometric meaning. Consider a congruence of lines for which the vector of the tangent $\mathbf{a}$ is defined by the formula

$$\mathbf{a} = [B \operatorname{grad} u,\ B \operatorname{grad} v]/(B \operatorname{grad} u,\ B \operatorname{grad} u),$$

where $u + iv = w$ and $(B \operatorname{grad} w,\ B \operatorname{grad} w) = 0$ (the square bracket in the numerator denotes the vector product). If the matrix B is the identity, then the congruence is called isotropic. In the general case it is natural to call the congruence isotropic with respect to the matrix B. If the components of the vector $\mathbf{a}$ are written in the form $\mathbf{a}(\cos\theta, \cos\Phi\sin\theta, \sin\Phi\sin\theta)$, then

$$\lambda = \ln \operatorname{tg}\theta/2 + i\Phi, \qquad \vec{\gamma} = \mathbf{a} - \vec{\delta}\cos\theta.$$

Let $\mathbf{n}$ and $\mathbf{b}$ be the vectors of the normal and binormal to the lines of the congruence. Then

$$\mathbf{n} + i\mathbf{b} = -\frac{1}{\rho(\mathbf{a}, \nabla)\lambda}\,\vec{\delta},$$

where ρ is the radius of curvature of the lines of the congruence. Condition (5), or, what is the same thing, (9), is the necessary and sufficient condition for isotropy of the congruence with respect to the matrix B. It has the form of a quasilinear equation of first order for the function λ. Its general solution will be $H = (\Psi_1, \Psi_2, \Psi_3) = 0$, where H is an arbitrary function, and $\Psi_k(x, \lambda)$ are independent solutions of the linear homogeneous equation

$$(\vec{B}\vec{\delta}, \nabla)\Psi + i(\vec{\delta}, T\vec{\delta})\partial\Psi/\partial x = 0.$$

In the case of an isotropic congruence (B is the identity), (9) is easily integrated, and we obtain a formula giving (in implicit form) the general form of isotropic congruences in three-dimensional space:

$$\lambda = H(x_1 \operatorname{sh}\lambda - x_2,\ x_1 \operatorname{ch}\lambda - ix_3),$$

where H is an arbitrary function.

Remark 2. If the elements of the matrix B and the vector Z are analytic, one can construct f.i. solutions analytic with respect to x_1, x_2, x_3, and thus arrive at f.i. solutions of the hyperbolic equation,

In this way nonanalytic solutions can also be obtained. Thus, for example, starting from an equation whose principal part is in the form of the Laplace operator, one can show that the equation

$$u_{xx} + u_{yy} - u_{tt} + \frac{k\cos\theta - G_\sigma \sin\theta}{1 - (t-\tau)G_\sigma - \sigma G_\tau} u_x + \frac{k\sin\theta + G_\sigma \cos\theta}{1 - (t-\tau)G_\sigma - \sigma G_\tau} u_y = 0,$$

where $\tau = x\sin\theta - y\cos\theta + t$; $\sigma = x\cos\theta + y\sin\theta$; $k(x, y, t)$, $G(\sigma, \tau)$ are arbitrary functions of their arguments; $\theta(x, y, t)$ is determined from the equation $\theta = G(\sigma, \tau)$, has a f.i. solution determined as the integral of the ordinary equation $d\tau/d\sigma = \sigma G_\sigma/(1 - \sigma G_\tau)$. If we take $G = G(\tau)$, $k = 0$, then we obtain the well-known Smirnov-Sobolev solutions for the wave equation.

Remark 3. The method presented in this paper can, with appropriate changes, be applied to the study of f.i. solutions of an elliptic equation with n independent variables. Relations (1), (2), (3) will also be satisfied in this case, but the components of the vector $\vec{\delta}$ will depend on $n-2$ complex functions λ_k. Instead of $\vec{\gamma}$ there will appear $n-2$ vectors $\vec{\gamma}_k = \vec{\delta}_{\lambda_k}$. Relation (5) will be replaced by $\frac{1}{2}(n-1)(n-2)$ conditions, whence it follows that in n-dimensional space ($n > 3$) f.i. solutions will not exist for arbitrary principal parts (1). Instead of the system (6) there will appear a system of $n-1$ equations, for which the conditions indicated above will be completeness conditions. Naturally, the equations for λ_k corresponding to (9), as well as the necessary and sufficient conditions for the existence of f.i. solutions linking the elements of the matrix B and the components of the vector Z, become more complicated. Finding f.i. solutions, when such conditions are satisfied, will reduce to solving the complete system of equations $(B\vec{\delta}, \nabla)w = 0$, $(B\vec{\gamma}_k, \nabla)w = 0$, which in turn reduces to the successive solution of $n-1$ ordinary first-order equations. The vectors $\vec{\delta}$, $\vec{\gamma}_k$ also have a simple geometric meaning in the n-dimensional case.

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CITED LITERATURE

[1] V. I. Smirnov, *Vestn. LGU*, No. 8 (1953).
[2] V. I. Smirnov, *Vestn. LGU*, No. 11 (1953).
[3] V. I. Smirnov, *Vestn. LGU*, No. 5 (1954).

Note: Figure translations are in progress. See original paper for figures.