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Abstract

Full Text

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MATHEMATICS

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COMPUTATION OF INTEGRALS OF INFINITELY DIFFERENTIABLE FUNCTIONS

In a previous note by the author ⁽¹⁾ it was shown that, in the class of periodic functions from $L_2^{(m)}$, the error estimate in computing an integral by means of the grid method, for each individual function, always turns out to be better than the estimate through the norm of the functional, although for the whole of $L_2^{(m)}$ this estimate through the norm is, of course, unimprovable. Considering countably normed spaces of infinitely differentiable functions, we encounter another similar circumstance. Each such space is the intersection of a countable sequence of spaces $L_2^{(m)}$, $m = 1, 2, \dots$. The cubature formulas for periodic functions with constant coefficients indicated in that note are valid for all spaces $L_2^{(m)}$ for arbitrary m , and for any function belonging to all of them we obtain a countable sequence of estimates:

$$(l, \varphi) \leq (h/2\pi)^m \sqrt{\xi(H^{-1}, 2m)} \sqrt{\Omega} \|\varphi\|_{L_2^{(m)}}, \quad m = [n/2] + 1, [n/2] + 2, \dots \quad (1)$$

Having specified some law of growth of $L_2^{(m)}$, we can choose, for each given h , the best estimate as the lower bound of all (1). It turns out that in a number of cases one can in this way establish the convergence of (l, φ) to zero as $h \rightarrow 0$ much faster than any power. The present note is devoted to this question.

Lemma 1. For large values of m , the function $\xi(A, m)$ admits the estimate

$$\xi(H^{-1}, 2m) \leq \frac{K_{\min}}{r_{\min}^{2m}} (1 + o(1)), \quad (2)$$

where r_{\min} is the shortest distance between the points of the integer lattice βH^{-1} , β is an integer row vector, and K_{\min} is the number of such points that

are at this shortest distance from the given point.

The proof of this lemma follows trivially from the definition of the function $\xi(H^{-1}, 2m)$ (see (1)).

We shall consider classes of real infinitely differentiable periodic functions for which derivatives of order α are subject to the inequalities

$$|D^\alpha \varphi / \alpha!| < KA^{|\alpha|} |\alpha|^{(\beta-1)|\alpha|}. \quad (3)$$

Here, for brevity, $\alpha!$ denotes $\alpha! = \alpha_1! \cdots \alpha_n!$; the remaining notation is the same as in (1, 2).

The number β will be called the **order of growth** of the derivatives of φ , and the number A the **type** of this growth. When (3) is fulfilled, we shall write

$$\varphi \in \mathfrak{R}(A, \beta). \quad (4)$$

For functions of several independent variables, almost without any changes, the basic properties of such classes, long established for functions of one independent variable, carry over. Let us recall these properties.

One may distinguish five cases, not all of which are of interest: a) $\beta < 0$; b) $\beta = 0$; c) $0 < \beta < 1$; d) $\beta = 1$; e) $\beta > 1$.

In case a), the class $\mathfrak{R}(A, \beta)$ contains no periodic functions except the constant function, and we shall not consider it.

In case b), i.e., for $\beta = 0$, the class $\mathfrak{R}(A, \beta)$ of periodic functions can contain only polynomials whose degree depends on the value of the constant A : $n \leq K_1 A$.

In case c), for $0 < \beta < 1$, $\mathfrak{R}(A, \beta)$ contains entire functions of order of growth

$$\rho = \frac{1}{1 - \beta} \quad (5)$$

and of type

$$\sigma = (1 - \beta)A^{1/(1-\beta)}/e. \quad (6)$$

In case d), for $\beta = 1$, the class $\mathfrak{R}(A, \beta)$ consists of functions analytic with radius of convergence at each point determined by the constant A ,

$$\text{Im}\{x_j\} < e^{-K_2 A}, \quad (7)$$

where K_2 is some constant.

Finally, for $\beta > 1$ the class \mathfrak{K} consists of infinitely differentiable nonanalytic functions. For any finite A , it contains finite functions. Various quasi-analytic functions are contained in the intersection $\bigcap_{A>0} \mathfrak{K}(A, 1)$.

Lemma 2. *A periodic function φ of the variable x in n -dimensional space, belonging to the class $\mathfrak{K}(A, \beta)$, has the following estimate for the norm $L_2^{(m)}$, $m = 0, 1, 2, \dots$:*

$$\|\varphi\|_{L_2^{(m)}} \leq K_3 m^{\beta m + 1/2} (A/e)^m, \quad (8)$$

where K_3 is some constant depending on K_1 and Ω .

The proof of this lemma is based on simple estimates. Recall that we shall always understand $\sum_{|\alpha|=m} f(\alpha)$ to denote such a summation where the function $f(\alpha)$ may also not have symmetry with respect to permutations of the indices in the vector $(\alpha_1, \alpha_2, \dots, \alpha_n)$ on which it depends. If such symmetry exists, then the sum is taken with the corresponding repetitions. The norm of the function $\|\varphi\|_{L_2^{(m)}}$ is written as

$$\|\varphi\|_{L_2^{(m)}}^2 = \int \sum_{|\alpha|=m} (D^\alpha \varphi)^2 dx = \int \sum_{|\alpha|=m} \left(\alpha! \frac{D^\alpha \varphi}{\alpha!} \right)^2 dx. \quad (9)$$

Since

$$\sum_{|\alpha|=m} \left(\alpha! \frac{D^\alpha \varphi}{\alpha!} \right)^2 \leq A^{2m} m^{2m(\beta-1)} \sum_{|\alpha| \leq m} (\alpha!)^2, \quad (10)$$

we obtain

$$\|\varphi\|_{L_2^{(m)}} \leq A^m m^{m(\beta-1)} \sqrt{\Omega} \left[\sum_{|\alpha|=m} (\alpha!)^2 \right]^{1/2}. \quad (11)$$

The inequality

$$\sum_{|\alpha|=m} (\alpha!)^2 \leq K e^{-2m} m^{2m+1}, \quad (12)$$

is valid, proved by regrouping the sum

$$\sum_{|\alpha|=m} (\alpha!)^2 = m! \sum_{|\alpha^{(j)}|=m} (\alpha^{(j)})!, \quad (13)$$

where the sum on the right-hand side is taken without repetitions, only over distinct integer-valued vectors $\alpha^{(j)}$. Next one establishes

$$\sum_{|\alpha|=m} \frac{(\alpha^{(j)})!}{(m-1)!} = n + O\left(\frac{1}{m}\right). \quad (14)$$

From (13) and (14), (12) follows at once. Finally, using Stirling's formula as applied to (11) and (12), we obtain the proof of Lemma 2.

Lemmas 1 and 2 make it possible to prove the main theorem.

Theorem 1. *For any function of the class $\mathfrak{F}(A, \beta)$, $\beta > 0$, the following asymptotic estimate holds for the error of the cubature formula with nodes at the points $hH\gamma$ as $h \rightarrow 0$:*

$$|(l, \varphi)| \leq Kh^{-1/2} \exp \left[-\frac{\beta}{e} \left(\frac{Ah}{2\pi e r_{\min}} \right)^{-1/\beta} \right]. \quad (15)$$

The proof of this theorem is based on finding the minimum with respect to m of the function on the right-hand side of the inequality:

$$\|l\|_{L_2^{(m)*}} \|\varphi\|_{L_2^{(m)}} = (h/2\pi)^m \sqrt{\Omega} \sqrt{\zeta(H^{-1}, 2m)} K^{m\beta+1/2} (A/e)^{1/2}, \quad (16)$$

which is not difficult to carry out elementarily.

Along the way it is established which m is optimal for a given value of h , namely

$$m = (Ah/2\pi e r_{\min})^{-1/\beta} e^{-1}. \quad (17)$$

It is curious to note a case that does not follow directly from this theorem, namely when $\beta = 0$. A direct estimate in this case gives Theorem 2.

Theorem 2. *For sufficiently small h , the cubature formula with a regular lattice is exact for any trigonometric polynomial.*

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CITED LITERATURE

1. S. L. Sobolev, DAN, 162, No. 6 (1965).
2. S. L. Sobolev, DAN, 162, No. 5 (1965).

Note: Figure translations are in progress. See original paper for figures.

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