ULTRARELATIVISTIC EXPANSION OF A GAS IN A GRAVITATIONAL FIELD

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Abstract

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In connection with observations of superstars and explosions of galactic nuclei, it is of interest to consider the expansion of a gas in the general theory of relativity when the velocity of the gas is very close to the speed of light. Such a problem is also of independent interest as an example of posing the Cauchy problem for Einstein' s equations.

The expansion of an ultrarelativistic gas in the special theory of relativity was considered in the works of L. D. Landau ([1]), I. M. Khalatnikov ([2]), K. P. Stanyukovich ([3]), and others, in connection with the problem of multiple production of particles in collisions of superfast nucleons. To use the methods developed by the above authors, one should write the equations of gas motion in the general theory of relativity in a form similar to the equations in the special theory of relativity. The latter is achieved by choosing the centrally symmetric interval

$$ds^2 = e^{\nu} d\tau^2 - r^2 d\Omega^2 - e^{\lambda} dr^2 \tag{1}$$

and writing the equations of motion and continuity in the form ([3,4]):

$$\frac{1}{\theta^2}\left(A\frac{\partial a}{\partial \tau} + a\frac{\partial a}{\partial r}\right) + \left(\frac{\partial \ln w}{\partial r} + aA\frac{\partial \ln w}{\partial \tau}\right) + \frac{1}{2}\left(aA\frac{\partial \lambda}{\partial \tau} + \frac{\partial \nu}{\partial r}\right) = 0,$$

$$\frac{1}{\theta^2}\left(Aa\frac{\partial a}{\partial \tau} + \frac{\partial a}{\partial r}\right) - \left(A\frac{\partial \ln v}{\partial \tau} + a\frac{\partial \ln v}{\partial r}\right) + \frac{2a}{r} + \frac{1}{2}\left(A\frac{\partial \lambda}{\partial r} + a\frac{\partial \nu}{\partial r}\right) = 0. \tag{2}$$

Here v is the specific volume, $w = v(p + \varepsilon)$ is the heat content per unit mass; a is the root-mean-square velocity measured in proper time; $\theta^2 = 1 - a^2$; $A = \exp[(\lambda - \nu)/2]$; the flow is assumed to be isentropic, i.e. $\sigma = \text{const}$. In expressions (1) and (2) we have put $c = 1$; here a is dimensionless and the time τ has the dimension of length.

The system (2) is supplemented by the Einstein field equations

$$e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2}=\chi\frac{(p+\varepsilon a^2)}{\theta^2};\qquad e^{-\lambda}\left(\frac{\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2}=\chi\frac{(\varepsilon+a^2p)}{\theta^2}\quad(3)$$

or

$$Ae^{-\lambda}\frac{\partial\lambda}{\partial\tau}=-\frac{\chi(p+\varepsilon)ar}{\theta^2}.$$

For an ultrarelativistic velocity, when

$$1-a=2\Delta,\qquad \Delta\ll 1,\quad(4)$$

and the equation of state is $p=(k-1)\varepsilon$ (below, in order to shorten the formulas, we put $k=4/3$, although other values of k do not complicate the solution), we shall—

···to have a considerably simpler initial system. The initial distribution of pressure and velocity (the Cauchy problem) should likewise be chosen in a simplified way, as is done, for example, in the self-similar expansion of an ordinary gas ([5]).

We shall assume that at the initial instant of time ($\tau=0$) there is a gas sphere of radius R, of constant pressure p_0, in which a constant velocity Δ_0, directed away from the center, is specified. Since in centrally symmetric motion there is no free gravitational field, the prescribed distribution of matter also determines, at the initial instant of time, the metric

$$\lambda=-\ln\left(1+\frac{\varkappa}{r}\int_0^r T_0^0r^2\,dr\right),\qquad \lambda+\nu=\varkappa\int_0^r(T_1^1-T_0^0)\,re^{\lambda}\,dr+C.\quad(5)$$

Here $\varkappa$ is Einstein' s constant; C is a constant of integration, determined from the condition that at the initial instant of time, at $r=R$, $\lambda+\nu=0$, as follows for the Schwarzschild metric. From (5) it then follows that

$$e^{-\lambda}=1-\frac{\varkappa p_0r^2}{3\Delta_0},\qquad e^{\nu}=\left(1-\frac{\varkappa p_0R^2}{3\Delta_0}\right)^3e^{2\lambda},\quad(6)$$

$$e^{-\lambda}=1-\frac{\varkappa p_0R^3}{3\Delta r}\quad\text{for } r<R;\qquad e^{\nu}=e^{-\lambda}\quad\text{for } r>R.$$

The ultrarelativistic approximation, after a number of transformations, makes it possible to obtain from systems (2), (3) the equations

$$A\frac{\partial p}{\partial \tau}+\frac{\partial p}{\partial r}=-\frac{4p}{r}, \qquad A\frac{\partial \lambda}{\partial \tau}+\frac{\partial \lambda}{\partial r}=-\left(\frac{e^{\lambda}-1}{r}+\varkappa r p e^{\lambda}\right),$$

$$A\frac{\partial \psi}{\partial \tau}+\frac{\partial \psi}{\partial r}=\frac{1}{2p}\left(A\frac{\partial p}{\partial \tau}+\frac{\partial p}{\partial r}\right), \qquad \psi=\ln\Delta+\lambda. \tag{7}$$

The system obtained is convenient in that the unknown ν enters it only through

$$A=\exp[(\lambda-\nu)/2],$$

and eliminating ν makes it possible to find a solution of part of the equations without resorting to those not written out in (7).

Elimination of A is achieved by passing to new independent variables p and r. The first equation of (7) then has the form

$$A-\frac{\partial \tau}{\partial r}=-\frac{4p}{r}\frac{\partial \tau}{\partial p}. \tag{8}$$

With the aid of (8), for the two other equations (7) we have

$$\frac{\partial \psi}{\partial r}-\frac{4p}{r}\frac{\partial \psi}{\partial p}=-\frac{2}{r}, \qquad e^{-\lambda}\left(\frac{\partial \lambda}{\partial r}-\frac{4p}{r}\frac{\partial \lambda}{\partial p}\right)=-\left(\frac{1-e^{-\lambda}}{r}+\varkappa r p\right). \tag{9}$$

For the first integrals of equations (9) we have the expressions

$$C_{1\lambda}=pr^4, \qquad C_{2\lambda}=r(1-e^{-\lambda})-\varkappa p r^3,$$

$$C_{1\psi}=pr^4, \qquad C_{2\psi}=\Delta e^{\lambda}p^{-1/2}. \tag{10}$$

It is directly evident from this that in the expansion of an ultrarelativistic gas in a gravitational field the pressure p falls as r^{-4}, i.e., in the same way as in the special theory of relativity. The solutions for λ and Δ satisfying the initial conditions specified above are found with the aid of (10) by the standard method*

$$e^{-\lambda}=1-b\left(\frac{p}{p_0}\right)^{3/4}r^2, \qquad \Delta=\Delta_0\left[\frac{1-b(p/p_0)^{3/4}r^2}{1-b(p/p_0)^{1/2}r^2}\right]\left(\frac{p}{p_0}\right)^{1/2}, \qquad b=\frac{\varkappa p_0}{3\Delta_0}. \tag{11}$$

* In obtaining (11), $\varkappa p r^3$, which is small compared with the terms containing b, has been omitted.

Thus, in the expansion of the gas the metric tends to the Euclidean one as $(p/p_0)^{1/4}$. In the absence of gravitational forces the velocity Δ, as follows from (11), would grow as $(p/p_0)^{1/2}$. Hence the quantity standing in square brackets determines the character of the slowing down of the velocity in the gravitational field.

To determine ν from the first two equations of system (3), we have

$$2A\frac{\partial\lambda}{\partial p} = -\frac{\partial\tau}{\partial p}\frac{\partial\varphi}{\partial r} + \frac{\partial\tau}{\partial r}\frac{\partial\varphi}{\partial p}, \qquad \varphi = \lambda + \nu. \tag{12}$$

Eliminating from (12), with the help of the third equation of (3) and (8), $\partial\tau/\partial r$ and $\partial\tau/\partial p$, we obtain

$$2\frac{\partial\lambda}{\partial p} - \frac{\partial\varphi}{\partial p} = e^{-\lambda}\frac{\partial\lambda}{\partial p}\frac{\Delta}{\chi p r}\left(\frac{\partial\varphi}{\partial r} - \frac{4p}{r}\frac{\partial\varphi}{\partial p}\right). \tag{13}$$

In the last equation λ and Δ have already been found. Determining from (13) $\nu = \varphi - \lambda$, we find with the aid of (8) $\tau = \tau(r, p)$. Such is the scheme for solving the problem, first proposed in ([6]).

Because of its unwieldiness, it is not possible to find the solution of (13) in analytic form, which makes it necessary to find ν and τ by numerical integration.

There is, however, a comparatively simple way of clarifying the whole picture of the motion. Let us first consider the expansion of an ultrarelativistic gas in an external Schwarzschild field. Such a problem describes the expansion of the envelope of a relativistic star after the shock wave coming from the center reaches the surface. The mass of the expanding gas, in comparison with the mass of the remaining star M, will be negligibly small. The gas therefore moves in the external Schwarzschild field created by the star,

$$e^{-\lambda} = e^{\nu} = (1 - r_g/r), \qquad r_g = 2kM/c^2. \tag{14}$$

If at $\tau = 0$ in an envelope of radius R the gas pressure was p_0, and the velocity was Δ_0, then for the velocity of the subsequent expansion at $r > R$, from (10) with the aid of (14) we have:

$$\Delta = \Delta_0\left[\frac{1 - r_g/r}{1 - r_g/R}\right]\left(\frac{p}{p_0}\right)^{1/2}, \tag{15}$$

and for the time τ from (8), with $A = (1 - r_g/r)^{-1}$, it follows that

$$\tau = r\left[1 - \left(\frac{p}{p_0}\right)^{1/4}\right] + r_g \ln\left[\frac{r - r_g}{r(p/p_0)^{1/4} - r_g}\right]. \tag{16}$$

As r_g tends to zero, from (15) and (16) there follows the known result for the expansion of a gas in the special theory of relativity,

$$\Delta = \Delta_0 \left(\frac{p}{p_0}\right)^{1/2}, \qquad \left(\frac{p}{p_0}\right) = \left(\frac{r-\tau}{r}\right)^4.$$

Let us now return to the original problem of the expansion of a gas sphere in its own gravitational field. In view of the fact that the distribution of matter inside the sphere affects the metric outside it only integrally, we shall assume, in accordance with (10), that for $r > R$, $(p/p_0) \sim (R/r)^4$; then for the velocity (11) we shall have

$$\Delta \cong \Delta_0 \left[\frac{1 - bR^3/r}{1 - bR^3/R}\right] \left(\frac{p}{p_0}\right)^{1/2}, \tag{17}$$

which coincides with (15). It follows from this that the ultrarelativistic expansion of an initially homogeneous gas sphere of radius R at $r > R$ proceeds effectively in the same way as the motion of gas in an external Schwarzschild field with gravitational radius $r_g = r_g^* = \chi p_0 R^3/3\Delta_0$; in this case $e^{-\lambda} = (1 - r_g^*/r)$.

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Note: Figure translations are in progress. See original paper for figures.