ON THE STABILITY OF PARALLEL FLOWS OF NON-NEWTONIAN MEDIA

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Abstract

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ON THE STABILITY OF PARALLEL FLOWS OF NON-NEWTONIAN MEDIA

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In the linear theory of stability of parallel flows of incompressible Newtonian fluids, Squire' s theorem ([1,2]) holds, according to which the problem of the stability of a flow with respect to three-dimensional periodic disturbances is equivalent to the problem of stability with respect to two-dimensional disturbances of flows with smaller Reynolds numbers. Squire' s theorem, in the linear theory of stability of parallel flows of incompressible non-Newtonian media, does not hold in general. It is shown below that the non-Newtonian properties of the medium, in particular, may have no influence whatever on stability with respect to two-dimensional disturbances and at the same time may substantially affect the behavior of three-dimensional disturbances.

Let us consider an isotropic incompressible medium whose motion is described by the equations

$$\rho\, du_j/dt = -\partial p_m/\partial y_j + \partial s_{kj}/\partial y_j + F_j, \qquad \varepsilon_{kk} = 0,$$

$$s_{kj} = s_{kj}(\varepsilon_{kj}, \dot{\varepsilon}_{kj}, \dot{s}_{kj}), \qquad p_m = p + \psi \quad (k, j = 1, 2, 3), \tag{1}$$

where ρ is the density; u_j are the projections of the velocity vector; p_m is the mean isotropic stress; y_j are Cartesian coordinates; F_j are the projections of the body force; s_{kj}, ε_{kj} are, respectively, the components of the deviators of the stress and strain-rate tensors; $\dot{s}_{kj}$, $\dot{\varepsilon}_{kj}$ are their time derivatives in one sense or another ([3]); ψ is a known function of the invariants of the tensors ε_{kj} and $\dot{\varepsilon}_{kj}$; p is an undetermined pressure.

Suppose that equations (1), for given body forces and boundary conditions, admit the solution $\mathbf{u}^0 = \{u_1^0(y_2), 0, 0\}$, p^0, corresponding to a parallel flow of the medium. We consider the stability of this solution with respect to three-dimensional disturbances, each component of which in the coordinate system

y_1, y_2, y_3 has the form

$$q'_j = \tilde{q}^*_j(y_2)\exp[i\tilde{\alpha}(y_1 - \tilde{c}t) + i\tilde{\beta}y_3],$$

where q'_j is a small disturbance; $\tilde{q}^*_j$ is the amplitude of the disturbance; $\tilde{c} = \tilde{c}_r + i\tilde{c}_i$, $j = 1, 2, 3$; $\tilde{c}_r, \tilde{c}_i, \tilde{\alpha}, \tilde{\beta}$ are real numbers. We note that in the coordinate system

$$x_1 = y_1\cos\theta + y_3\sin\theta, \qquad x_2 = y_2, \qquad x_3 = y_3\cos\theta - y_1\sin\theta,$$

where $\theta = \operatorname{arctg}(\tilde{\beta}/\tilde{\alpha})$, the stability problem under consideration is equivalent to the problem of stability of the flow with velocity field

$$\mathbf{u}^0 = \{u_1^0(x_2)\cos\theta,\ 0,\ -u_1^0(x_2)\sin\theta\}$$

with respect to disturbances of the form

$$q'_j = q^*_j(x_2)\exp[i\alpha(x_1 - ct)], \qquad j = 1, 2, 3.$$

In the coordinates x_1, x_2, x_3 the linearized equations of motion are written in the form

$$\rho\left(\frac{\partial u'_1}{\partial t} + \frac{du_1^0}{dx_2}u'_2\cos\theta + \frac{\partial u'_1}{\partial x_1}u_1^0\cos\theta\right) = -\frac{\partial p'_m}{\partial x_1} + \frac{\partial s'_{11}}{\partial x_1} + \frac{\partial s'_{12}}{\partial x_2}, \qquad \frac{\partial u'_1}{\partial x_1} + \frac{\partial u'_2}{\partial x_2} = 0,$$
$$\rho\left(\frac{\partial u'_2}{\partial t} + \frac{\partial u'_2}{\partial x_1}u_1^0\cos\theta\right) = -\frac{\partial p'_m}{\partial x_2} + \frac{\partial s'_{12}}{\partial x_1} + \frac{\partial s'_{22}}{\partial x_2},$$
$$\rho\left(\frac{\partial u'_3}{\partial t} + \frac{\partial u'_3}{\partial x_1}u_1^0\cos\theta - u'_2\frac{du_1^0}{dx_2}\sin\theta\right) = \frac{\partial s'_{13}}{\partial x_1} + \frac{\partial s'_{23}}{\partial x_2}. \tag{2}$$

where the perturbations p'_m, s'_{kj}, u'_j satisfy the system of equations obtained by linearizing the last two relations (1) in the coordinate system x_1, x_2, x_3.

Eliminating p'_m from the first two equations of motion (2), we obtain

$$\rho\left[\frac{\partial^2 u'_1}{\partial t\,\partial x_2} - \frac{\partial^2 u'_2}{\partial t\,\partial x_1} + u'_2\frac{d^2u_1^0}{dx_2^2}\cos\theta + u_1^0\frac{\partial}{\partial x_1}\left(\frac{\partial u'_1}{\partial x_2} - \frac{\partial u'_2}{\partial x_1}\right)\cos\theta\right] = \frac{\partial^2 s'_{11}}{\partial x_1\,\partial x_2} + \frac{\partial^2 s'_{12}}{\partial x_2^2} - \frac{\partial^2 s'_{12}}{\partial x_1^2} - \frac{\partial^2 s'_{22}}{\partial x_1\,\partial x_2}. \tag{3}$$

For an incompressible Newtonian fluid ($s'_{kj} = 2\mu e'_{kj}$), the right-hand side of equation (3) does not contain u'_3; moreover, (3) coincides with the equation for two-dimensional perturbations of a parallel flow with velocity $u_1^0\cos\theta$. For a non-Newtonian medium the right-hand side of (3) may contain u'_3, and Squire' s theorem may fail to hold.

As an example, let us consider the stability of a layer of a Reiner–Rivlin fluid flowing under the action of gravity down an inclined plane. We write the defining equations of the medium in the form [4]

$$\varepsilon_{kk} = 0, \qquad \psi = -\frac{\mu_1}{3}\varepsilon_{kj}\varepsilon_{kj}, \qquad s_{kj} = \psi\delta_{kj} + 2\mu\varepsilon_{kj} + 2\mu_1\varepsilon_{kl}\varepsilon_{lj} \quad (l, k, j = 1, 2, 3), \tag{4}$$

where μ, μ_1 are constant coefficients of Newtonian and transverse viscosity, and δ_{kj} is the Kronecker symbol.

Equations (1), for $\mathbf{F} = \{\rho g \sin\gamma, \rho g \cos\gamma, 0\}$, have the exact solution

$$u_1^0 = \frac{\rho g \sin\gamma}{2\mu}(d^2 - y_2^2), \qquad p^0 = \rho g y_2 \cos\gamma + \frac{\mu_1 \rho^2 g^2 \sin^2\gamma}{2\mu^2} y_2^2, \qquad u_* = \frac{\rho g d^2 \sin\gamma}{3\mu},$$
$$\sigma_{12}^0 = -\rho g y_2 \sin\gamma, \qquad \sigma_{23}^0 = \sigma_{13}^0 = 0, \qquad \sigma_{33}^0 = -p^0, \qquad \sigma_{11}^0 = \sigma_{22}^0 = -\rho g y_2 \cos\gamma,$$
$$\sigma_{kj}^0 = -p_m^0 \delta_{kj} + s_{kj}^0 \qquad (k, j = 1, 2, 3), \tag{5}$$

which describes the flow of a layer of thickness d of the fluid under consideration, flowing with mean velocity u_* under the action of the gravitational force ρg down a plane inclined to the horizontal at the angle γ. In (5), the axis y_1 is directed along the free surface at the angle γ to the horizontal, and the axis y_2 is directed into the layer. On the free surface $y_2 = 0$ it is assumed that $\sigma_{12}^0 = 0$, $\sigma_{22}^0 = 0$, and on the plane $y_2 = d$ the condition $u_1^0 = 0$ is imposed.

Keeping the previous notation, we pass to dimensionless variables by referring quantities with the dimension of length to the quantity d, quantities with the dimension of velocity to the quantity u_*, time t to the quantity d/u_*, and all stresses to the quantity ρu_*^2.

Introduce into the coordinate system x_1, x_2, x_3 the stream function φ by the relations $u_1' = \partial\psi/\partial x_2$, $u_2' = -\partial\psi/\partial x_1$, putting $\psi = \varphi(x_2)\exp[i\alpha(x_1 - ct)]$, $u_3' = u_3^*(x_2)\exp[i\alpha(x_1 - ct)]$. Equation (3) and the last of equations (2), in dimensionless variables, take the form

$$\frac{d^4\varphi}{dx_2^4} - 2\alpha^2\frac{d^2\varphi}{dx_2^2} + \alpha^4\varphi = i\alpha\,\mathrm{Re}\left[(u_1^0\cos\theta - c)\left(\frac{d^2\varphi}{dx_2^2} - \alpha^2\varphi\right) - \frac{d^2u_1^0}{dx_2^2}\varphi\cos\theta\right] + \frac{3i\alpha x_2\,\mathrm{Re}}{2\,\mathrm{Re}_1}\left(\alpha^2 u_3^* - \frac{d^2u_3^*}{dx_2^2}\right)\sin\theta,$$

$$i\alpha\left[u_3^*(u_1^0\cos\theta - c) + \varphi\frac{du_1^0}{dx_2}\sin\theta\right] = \frac{1}{\mathrm{Re}}\left(\frac{d^2u_3^*}{dx_2^2} - \alpha^2u_3^*\right) + \frac{3}{\mathrm{Re}_1}\left[\frac{i\alpha x_2}{2}\left(\alpha^2\varphi - \frac{d^2\varphi}{dx_2^2}\right)\sin\theta - i\alpha\frac{d\varphi}{dx_2}\sin\theta - \frac{i\alpha}{2}u_3^* \right. \tag{6}$$

where $\mathrm{Re} = u_* d\rho/\mu$, $\mathrm{Re}_1 = \rho d^2/\mu_1$, $u_1^0 = {}^3/{}_2(1 - x_2^2)$.

Linearizing the boundary conditions in stresses on the perturbed free surface $x_2 = \eta(x_1, t)$, taking into account the kinematic condition (5) and the first of equations (2), we obtain the boundary conditions for equations (6) in the form

$$\frac{d^2\varphi(0)}{dx_2^2} + \left(\alpha^2 - \frac{3\cos\theta}{c^*}\right)\varphi(0) = 0,$$

$$\frac{\alpha}{c^*}(3\operatorname{ctg}\gamma + \alpha^2 S\,\mathrm{Re})\varphi(0) + \alpha(\mathrm{Re}\,c^* + 3i\alpha)\frac{d\varphi(0)}{dx_2} - i\frac{d^3\varphi(0)}{dx_2^3} - \frac{3}{2}\frac{\mathrm{Re}}{\mathrm{Re}_1}u_3^*\alpha\sin\theta = 0,$$

$$\frac{du_3^*(0)}{dx_2} = -\frac{3\varphi(0)\sin\theta}{c^*}, \qquad \frac{d\varphi(1)}{dx_2} = \varphi(1) = u_3^*(1) = 0 \tag{7}$$

$$\left(S = T/\rho u_*^2 d, \quad c^* = c - {}^3/{}_2\cos\theta\right),$$

where T is the coefficient of surface tension.

To determine the sought quantities φ, c, u_3^*, we use the method of successive approximations (5), representing φ, c, u_3^* as series

$$\varphi = \varphi^0 + \alpha\varphi' + \alpha^2\varphi'' + \cdots, \qquad c = c^0 + \alpha c' + \alpha^2 c'' + \cdots, \tag{8}$$

$$u_3^* = u_3^{*0} + \alpha u_3^{*'} + \alpha^2 u_3^{*''} + \cdots$$

Linearizing (6) and (7) with respect to the small parameter α, we obtain systems of equations of successive approximations.

To determine the quantities φ^0, c^0, u_3^{*0} we have the relations

$$\frac{d^4\varphi^0}{dx_2^4} = 0, \qquad \frac{d^2u_3^{*0}}{dx_2^2} = 0, \qquad \frac{d^3\varphi^0(0)}{dx_2^3} = \varphi^0(1) = \frac{d\varphi^0(0)}{dx_2} = u_3^{*0}(1) = 0,$$

$$\frac{du_3^{*0}(0)}{dx_2} - \frac{3\sin\theta\cdot\varphi^0(0)}{c^{*0}}, \qquad \frac{d^2\varphi^0(0)}{dx_2^2} - \frac{3\cos\theta}{c^{*0}}\varphi^0(0) = 0, \qquad c^{*0} = c^0 - {}^3/{}_2\cos\theta. \tag{9}$$

From (9) we find, up to a constant multiplier,

$$\varphi^0 = (1 - x_2)^2, \qquad c^0 = 3\cos\theta, \qquad u_3^{*0} = 2(1 - x_2)\operatorname{tg}\theta. \tag{10}$$

The equations for determining c', φ', taking (10) into account, can be written in the form

$$\frac{d^4\varphi'}{dx_2^4} = -6i \operatorname{Re} x_2 \cos\theta, \qquad \frac{d^2\varphi'(0)}{dx_2^2} = -\frac{4c'}{3\cos\theta}, \qquad \varphi'(1) = \frac{d\varphi'(1)}{dx_2} = 0,$$

$$\frac{2}{3\cos\theta}\left(3\operatorname{ctg}\gamma + \alpha^2 S \operatorname{Re}\right) - 3\operatorname{Re}\cos\theta - 3\frac{\operatorname{Re}}{\operatorname{Re}_1}\sin\theta \ \operatorname{tg}\theta - i\frac{d^3\varphi'(0)}{dx_2^3} = 0. \quad (11)$$

From (11) we easily find the expression of interest to us for c':

$$c' = i\operatorname{Re}\left(\frac{6}{5}\cos^2\theta - \frac{3}{2}\frac{\sin^2\theta}{\operatorname{Re}_1}\right) - \frac{i}{3}\left(3\operatorname{ctg}\gamma + \alpha^2 S\operatorname{Re}\right). \quad (12)$$

Restricting ourselves in (8) to the first two terms of the expansions, we obtain, setting $c_i = 0$, the equation of the neutral curve in the $\alpha, \operatorname{Re}$ plane:

$$\alpha = 0, \qquad \operatorname{Re}\left(\frac{6}{5}\cos^2\theta - \frac{3}{2}\frac{\sin^2\theta}{\operatorname{Re}_1}\right) - \frac{1}{3}\left(3\operatorname{ctg}\gamma + \alpha^2 S\operatorname{Re}\right) = 0. \quad (13)$$

The branching point of the neutral curve has coordinates

$$\alpha = 0, \qquad \operatorname{Re}^* = \frac{10\operatorname{Re}_1\operatorname{ctg}\gamma}{12\operatorname{Re}_1\cos^2\theta - 15\sin^2\theta}. \quad (14)$$

It is seen from (6) and (7) that the parallel flow of a Reiner–Rivlin fluid with respect to two-dimensional disturbances ($\theta = 0$) is stable or unstable in the same way as the flow of an incompressible Newtonian fluid of density ρ and viscosity μ. The transverse viscosity μ_1 affects the form of the neutral curve (13) and the position of the branching point (14) only for three-dimensional disturbances.

It follows from (14) that for $\theta = 0$ at the branching point $\operatorname{Re}_0^* = \frac{5}{6}\operatorname{ctg}\gamma$, while for $\theta \neq 0$, $\operatorname{Re}_1 = \infty$, $\mu_1 = 0$, at the branching point $\operatorname{Re}_\theta^* = 5\operatorname{ctg}\gamma/6\cos^2\theta$.

Without dwelling on all the consequences following from (12)–(14), we note that for $\mu_1 > 0$, for any $\theta \neq 0$, at small α the region of stability in the $\alpha, \operatorname{Re}$ plane is larger than for $\theta = 0$. If one assumes that the basic flow (5) is possible when $p_m^0 = \operatorname{Re}^{-1}\operatorname{ctg}\gamma x_2 + 3\operatorname{Re}_1^{-1} x_2^2 \geq 0$, then for $\mu_1 < 0$ the parallel flow of the layer is possible for $|\operatorname{Re}_1|\operatorname{ctg}\gamma \geq \operatorname{Re}$. In this case, for $|\operatorname{Re}| > 5/4$, from (14) we find that $\operatorname{Re}_\theta^* > \operatorname{Re}^*$ for any $\theta \neq 0$, and, consequently, in the $\alpha, \operatorname{Re}$ plane the region of stable three-dimensional waves is smaller than that of two-dimensional waves. Squire’ s theorem does not hold.

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Note: Figure translations are in progress. See original paper for figures.