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Abstract

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MATHEMATICS

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ON THE RUDIN-CARLESON THEOREM

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Rudin (1) and Carleson (2) proved the following remarkable theorem (see also (3)). If F is a closed subset of points of the circle $\Gamma : |\xi| = 1$, having measure zero, and $f(\xi)$ is an arbitrary continuous function on F, then there exists a function $\varphi(z)$, analytic in the disk $|z| < 1$, continuous in the closed disk $|z| \leq 1$, and coinciding on F with $f(\xi)$. Moreover, one may assume that $\max|\varphi(z)|$, for $|z| \leq 1$, coincides with $\max|f(\xi)|,\ \xi \in F$. An extension of this theorem to the case of functional algebras more general than the algebra of continuous analytic functions was given by Bishop (4). A profound strengthening of this theorem was obtained by Pelczyński (5,6), who also considered the space of continuous functions. We also point to a paper of Gamelin (7). In the present note we consider an analogue of the Rudin-Carleson theorem in the case of subspaces of an arbitrary Banach space X.

Theorem 1. *Let X be a Banach space, X^* its conjugate space, and suppose that the following conditions are fulfilled:*

1. $X^* = X_1^* \oplus X_2^*$, *and the subspace* X_2^* *is weakly closed.*
2. *If* $x_1^* \in X_1^*$ *and* $x_2^* \in X_2^*$, *then* $\|x_1^* + x_2^*\| \geq \|x_2^*\|$.
3. *A closed subspace* $Z \subset X$ *is such that, if* $x^*(z) = 0,\ z \in Z$, *then* $x^* \in X_1^*$.

Denote by X_2 *the annihilator of* X_2^* *in the space* X: $X_2 = (X_2^*)_\perp$, *and let* $Z_2 = Z \cap X_2$. *Then* $X/X_2 \simeq Z/Z_2$ *(*$\simeq$ *is an isomorphic isometry).*

Remark. Let us first of all explain why this theorem is a generalization of the Rudin-Carleson result. Let $X = C(\Gamma)$, the space of continuous complex-valued functions on Γ; let Z be the subspace of $C(\Gamma)$ consisting of functions admitting analytic continuation into the disk $|z| < 1$. The space X^* consists, as is known, of all finite Borel measures on Γ. Let X_2^* consist of all those measures whose support is contained in F, and X_1^* of the measures concentrated on $\Gamma \setminus F$. Condition 1 is here, obviously, fulfilled. Condition 2 is fulfilled in exactly the same way, since in our case

$$\|x_1^* + x_2^*\| = \|x_1^*\| + \|x_2^*\|.$$

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If some functional x^* is orthogonal to Z, then the measure defining it, by the F. and M. Riesz theorem, is absolutely continuous, and therefore its restriction to F is equal to zero, which means that condition 3 is fulfilled.

The subspace X_2 is composed of all functions equal to zero on F, and, as is easily seen, $X/X_2 \simeq C(F)$. The main assertion of the Rudin–Carleson theorem consists precisely in the fact that $Z/Z_2 \simeq C(F)$, and therefore follows from the theorem we have stated.

Proof of the theorem. For brevity denote $R = X/X_2$ and $R_1 = Z/Z_2$.

Define a mapping of R_1 into R in the following way. To the class $\alpha = \{z_0 + z_2\} \in R_1$, where z_0 is a fixed element of Z, and z_2 runs through Z_2, we put in correspondence the class $\tilde{\alpha} = \{z_0 + x_2\}$, where x_2 runs through X_2. The image of R_1 under our mapping will be denoted by $\widetilde{R}_1$. The mapping of R_1 onto $\widetilde{R}_1$ is one-to-one. Indeed, if $\alpha_1 \neq \alpha_2$, where $\alpha_1 = \{z_0' + z_2\}$ and $\alpha_2 = \{z_0'' + z_2\}$, then $z_0' - z_0'' \notin Z_2$, and therefore $z_0' - z_0'' \notin X_2$, and consequently—

indeed, $\tilde{\alpha}_1 \neq \tilde{\alpha}_2$. It is now clear that the mapping R_1 onto $\widetilde{R}_1$ is an isomorphism. Our next step will be to prove that $\widetilde{R}_1$ is dense in R. The annihilator $X_2^\perp$ of the subspace X_2 in the space X^* coincides with X_2^* by the weak closedness of $X_2^* : X_2^\perp = [(X_2^*)_\perp]^\perp = X_2^*$ (see, for example, (^{8}), Ch. II, § 9). By the Hahn–Banach theorem this allows us to identify R^* with X_2^*. Indeed, any functional $\varphi \in R^*$ may be regarded as defined on X and orthogonal to X_2; conversely, any $x_2^* \in X_2^*$ may be regarded as a functional on R, and it is also easy to see that if $\varphi \in R^*$ and $x_2^* \in X_2^*$ correspond to one another, then $\|\varphi\|_R = \|x_2^*\|_X$. This follows from the fact that, on the one hand, $|x_2^*(x)| = |\varphi(\beta)| = |\varphi(\{x + x_2\})| \leq \|\varphi\|_R \cdot \|\beta\| \leq \|\varphi\|_R \cdot \|x\|$, and hence $\|x_2^*\|_X \leq \|\varphi\|_R$; and, on the other hand,

$$\|\varphi\|_R = \sup_{\|\beta\| \leq 1} |\varphi(\beta)| = \sup_{\|x+x_2\| \leq 1} |x_2^*(x)| = \sup_{\|x+x_2\| \leq 1} |x_2^*(x+x_2)| \leq \sup_{\|t\| \leq 1} |x_2^*(t)| = \|x_2^*\|_X.$$

Thus, $R^* \simeq X_2^*$.

If now $\varphi \in R^*$ and $\varphi(\tilde{\alpha}) = 0$ for $\tilde{\alpha} \in \widetilde{R}_1$, then this means that the corresponding functional $x_2^* \in X_2^*$ vanishes on Z, and therefore, by assumption 3, $x_2^* \equiv 0$. Hence $\varphi \equiv 0$, and $\widetilde{R}_1$ is dense in R.

We shall now prove that $R_1 \simeq \widetilde{R}_1$. Since $R_1 = Z/Z_2$ is a Banach space, it will follow from the last isometry that $\widetilde{R}_1$ is complete; and this, together with the density of $\widetilde{R}_1$ in R, leads to $R_1 = R \simeq R_1$. Consider the space Z^*. By the Hahn—Banach theorem one may suppose that Z^* consists of the same functionals $x^* \in X^*$, but now

$$\|x^*\|_Z = \inf_{q^* \in Z^\perp} \|x^* - q^*\| \leq \|x^*\|.$$

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Here it is important to remember that condition 3 means the inclusion $Z^{\perp} \subset X_1^*$. Condition 2 now gives, for every functional x_2^*,

$$\|x_2^*\|_Z = \inf_{q^* \in Z^{\perp}} \|x_2^* - q^*\| \geq \|x_2^*\|,$$

and consequently $\|x_2^*\|_Z = \|x_2^*\|$. Thus, X_2^* may be regarded as isometrically embedded in Z^*. Denote the image of X_2^* under this embedding by $\widetilde{X}_2^*$. We shall show that $\widetilde{X}_2^*$ is weakly closed in Z^*. By the Krein–Šmulian theorem (see, for example, (^{9}), p. 465) it suffices to show that the intersection of $\widetilde{X}_2^*$ with an arbitrary closed ball S centered at the origin of Z^* is weakly closed. Let $y^* \in Z^*$ be a limit point of the set $\widetilde{X}_2^* \cap S$. This means that some generalized sequence $\{\tilde{x}_2^{*l}\}$ (l ranges over the set of indices defining the generalized sequence) of elements of $\widetilde{X}_2^* \cap S$ has y^* as its limit in the weak topology of the space Z^*. Let T be the closed ball in X^* of the same radius as S. The set $X_2^* \cap T$ is bicompact in the weak topology of X^*, and therefore the generalized sequence $\{x_2^{*l}\}$, contained in $X_2^* \cap T$ and corresponding to the sequence $\{\tilde{x}_2^{*l}\}$, has in $X_2^* \cap T$ a generalized limit point x_2^*. From the definition of the weak topologies in Z^* and X^* it follows that on Z the functionals y^* and x_2^* coincide, and the latter means that $y^* \in \widetilde{X}_2^*$. The closedness of $\widetilde{X}_2^*$ in the weak topology of Z^* is proved. Evidently, we have $(\widetilde{X}_2^*)_{\perp} = Z_2$ and $Z_2^{\perp} = \widetilde{X}_2^*$.

Now, for the coset $\alpha = \{z_0 + z_2\} \in R_1$ we obtain (again with the help of the Hahn–Banach theorem)

$$\|\alpha\| = \inf_{z_2 \in Z_2} \|z_0 + z_2\| = \sup_{\substack{x_2^* \in \widetilde{X}_2^* \\ \|x_2^*\| \leq 1}} |\tilde{x}_2^*(z_0)| = \sup_{\substack{x_2^* \in X_2^* \\ \|x_2^*\| \leq 1}} |x_2^*(z_0)|.$$

For the coset $\tilde{\alpha} = \{z_0 + x_2\} \in \widetilde{R}_1$, corresponding to α under the isomorphism of R_1 onto $\widetilde{R}_1$, we have

$$\|\tilde{\alpha}\| = \inf_{x_2 \in X_2} \|z_0 + x_2\| = \sup_{\substack{x_2^* \in X_2^* \\ \|x_2^*\| \leq 1}} |x_2^*(z_0)| = \|\alpha\|.$$

The isometry $R_1 \simeq \widetilde{R}_1$ is proved, and this completes the entire proof of the theorem.

Returning to the Rudin-Carleson theorem, let us note that from our theorem it follows only that the analytic function $\varphi(z)$, extending $f(\zeta)$, can be chosen so that

$$\max_{|z| \leqslant 1} |\varphi(z)| \leqslant \max_{\zeta \in F} |f(\zeta)| + \varepsilon,$$

where $\varepsilon > 0$ is arbitrary. The possibility of extending $f(\zeta)$ without increasing the norm does not follow directly from our theorem. In our terms, the possibility of

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such an extension would mean that for every $z \in Z$ there is an element $z_2^0 \in Z_2$ such that

$$\|z + z_2^0\| = \inf \|z + z_2\|, \quad z_2 \in Z_2.$$

If D is a closed subset of $\Gamma \setminus F$, then the function $f(\zeta)$ on F can be extended to Γ continuously in such a way that the extending function is equal to zero on D. If, however, one restricts oneself only to functions $\varphi(z)$ analytic for $|z| < 1$, then for every $\varepsilon > 0$ there is an extension $\varphi(z)$ such that $|\varphi(z)| < \varepsilon$, $z \in D$. These facts carry over to the abstract situation considered in the following way.

Theorem 2. *Suppose that, under the hypotheses of Theorem 1, the subspace* $Q \subset X_1^*$ *is closed in the weak topology of* X^* *and* $P = Q^\perp$. *Then the image of* P *under the natural homomorphism* $X \to X/X_2$ *coincides with all of* X/X_2. *In other words, for every* $x \in X$ *there exist* $x_2 \in X_2$ *and* $p \in P$ *such that* $x = x_2 + p$. *Moreover, for every* $z \in Z$ *and* $\varepsilon > 0$ *there exists* $z_2 \in Z_2$ *such that*

$$\sup_{q \in Q,\ \|q\| \leqslant 1} |q(z + z_2)| = \inf_{p \in P} \|z + z_2 + p\| < \varepsilon.$$

Proof. Let U be the operator mapping P into $X/X_2 = R$ under the natural homomorphism $X \to X/X_2$. To prove the first assertion of the theorem it is necessary, for the adjoint operator U^*, to establish the inequality

$$\|U^*(x_2^*)\| \geqslant m\|x_2^*\|$$

for some $m > 0$, since the space $R^* = X_2^*$. But

$$U^*(x_2^*) = x_2^* + q$$

with some $q \in P^\perp = Q$. Since $Q \subset X_1^*$, by assumption 2 of Theorem 1,

$$\|x_2^* + q\| \geqslant \|x_2^*\|.$$

To prove the second assertion of the theorem, we shall show that

$$\inf_{z_2 \in Z_2,\ p \in P} \|z + z_2 + p\| = \left\| \sup_{\|x^*\| \leqslant 1,\ x^* \in Z_2^\perp \cap Q} |x^*(z)| \right\| = 0.$$

Every functional $x^* \in Z_2^\perp$ has the form $x^* = x_2^* + z^*$, where $x_2^* \in X_2^*$, and $z^* \in Z^\perp \subset X_1^*$. Since $x^* \in Q$, we have $x_2^* + z^* = q \in Q \subset X_1^*$. Hence it follows that $x_2^* = 0$, and this completes the proof.

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