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Abstract

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MATHEMATICS

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ON THE INVARIANT EQUIPPING OF CERTAIN SURFACES OF SPECIAL TYPE IN PROJECTIVE SPACE

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Consider, in projective n-space P_n, an m-surface defined by the equation

$$\bar{x} = \bar{x}(\eta^a) \qquad (a, b, c, d, e = 1, \dots, m). \tag{1}$$

The points of the projective space P_n may be interpreted (see ([1])) as one-dimensional subspaces B_1 of a certain vector $(n+1)$-dimensional space B_{n+1}, called the associated vector space. In view of this, each point $\bar{x}$ of the surface (1) is simultaneously a contravariant vector of the associated space B_{n+1} and a basis vector of the corresponding one-dimensional vector subspace B_1. A renormalization of the vector $\bar{x}$ corresponds to a transformation of the basis in the one-dimensional vector space B_1, which is called the radial vector space.

The

$$\binom{m+v}{v}$$

vectors $\bar{x}, \bar{x}_a, \dots, \bar{x}_{a_1 \dots a_v}$ $\left(\bar{x}_{a_1 \dots a_u} = \partial^u \bar{x} / \partial \eta^{a_1} \cdots \partial \eta^{a_u}\right)$ of the associated space B_{n+1} define a vector subspace B_{k_v} of dimension k_v, which depends neither on the choice of admissible coordinates on the surface nor on the choice of basis in the radial B_1. This subspace is called the osculating space of order v. One says that an equipping of the surface (1) is given if, with each of its points, there is associated a decomposition of the space B_{n+1} into the direct sum $B_{n+1} = B_{m+1} + B_{n-m}$, where B_{m+1} is the osculating space of first order, and B_{n-m} is some subspace complementary to it, called the equipping space.

An m-surface in projective n-space will be called a **surface of special type** if all its osculating spaces, beginning with some order p satisfying the inequalities

$$2 \leq p, \qquad \binom{m+p}{p} \leq n+1,$$

do not have the maximum possible number of dimensions. The currently known schemes of invariant equipping, i.e. equipping determined by the surface (1) itself, are applicable to surfaces with the maximum possible dimensions of osculating spaces; these schemes cannot be extended in their entirety to surfaces of the special type indicated.

In the present article an invariant equipping is constructed for a surface of dimension m in projective n-space under the condition that the numbers m and n are related by

$$n = \binom{m+q-1}{q-1} - 1$$

(a) $q = 3,\ m = 3, 4$; b) $q > 3,\ m \geq 2$) and that the dimension of the osculating space of order $q - 1$ is decreased by one. To simplify the exposition, we shall restrict ourselves to the case $q = 4$.

Assume that the surface (1) is a surface of special type and that

$$n + 1 = \binom{m+3}{3}.$$

Let the $n + 1$ vectors $\bar{x}, \bar{x}_a, \bar{x}_{ab}, \bar{x}_{abc}$

of the attached space B_{n+1} are linearly dependent, but among them there are exactly n linearly independent ones. The indicated vectors determine, in an invariant way, an n-dimensional subspace of the attached B_{n+1}; the determining pseudocovector $\bar{\xi}(\xi_a)$ $(a = 1, \dots, n+1)$ of this subspace is uniquely found from the system

$$\bar{\xi}\bar{x} = 0, \qquad \bar{\xi}\bar{x}_a = 0, \qquad \bar{\xi}\bar{x}_{ab} = 0, \qquad \bar{\xi}\bar{x}_{abc} = 0, \tag{2}$$

which is a homogeneous system of rank n with respect to the components ξ_a of the pseudocovector $\bar{\xi}$.

Suppose now that the surface under consideration is partially framed with respect to the osculating space of third order B_{k_3}, i.e., for each point of the surface (1) a decomposition of the corresponding vector space B_{k_3} into the direct sum $B_{k_3} = B_{k_2} + \bar{H}_r$ is specified, where B_{k_2} denotes the osculating space of second order, and $\bar{H}_r$ denotes some vector subspace complementary to it of dimension

$$r = \binom{m+2}{3} - 1,$$

which is called a partially framing space with respect to B_{k_3}. Let $\bar{n}_i$ $(i, j = 1, \dots, r)$ be some fixed basis of the partially framing $\bar{H}_r$. Denoting by $\bar{l}_p$ $(p = 1, \dots, k_3 - r)$ some basis of the osculating space of second order B_{k_2}, we can represent each vector $\bar{x}_{abc}$ in the form

$$\bar{x}_{abc} = h^i_{abc}\bar{n}_i + T^p_{abc}\bar{l}_p. \tag{3}$$

The coefficients h^i_{abc} occurring in the expansion (3) are the components of a connecting affinor of three centro-affine spaces E_m, E_r, and E_1, respectively isomorphic to the tangent vector space $B^{(1)}_m$, the partially framing $\bar{H}_r$, and the radial B_1. The connecting affinor h^i_{abc} is a contravariant vector both in E_r and in E_1, and a covariant tensor in E_m. Under a change of the partially framing space $\bar{H}_r$, the object h^i_{abc} does not change and, consequently, is determined invariantly by the surface itself.

Suppose that the connecting affinor h^i_{abc} has a nonzero relative invariant I. The indicated invariant can be constructed, for example, as follows. Let Δ be a determinant of order $r+1$, composed in some ordered way of the essential components of the affinor h^i_{abc} and of some tensor X_{abc}. Expanding the determinant Δ by the elements of the row that contains only the components X_{abc}, we obtain the invariant equality $\mathfrak{A}^{abc}X_{abc} = \Delta$, where $\mathfrak{A}^{abc}$ is a connecting tensor density having weights $3(r+1)/m$, -1, and $-r$ with respect to the spaces E_m, E_r, and E_1, respectively. Consider the algebraic comitant I of the tensor density $\mathfrak{A}^{abc}$, equal to the discriminant of this density, i.e. $I = \mathrm{Dis}(\mathfrak{A}^{abc})$. Since the discriminant I is a homogeneous function of the components of the connecting tensor density $\mathfrak{A}^{abc}$ of degree $\lambda = mw/3$, where $w = 3 \cdot 2^{m-1}$, the weights of the connecting scalar density I of the spaces E_m, E_r, and E_1 are respectively wr, $-\lambda$, and $-\lambda r$. The existence of a nontrivial invariant I makes it possible (see ([2])) to introduce into consideration the affinor

$$h^{abc}_{\ \ \ i} = \frac{1}{I}\frac{\partial I}{\partial h^i_{abc}}, \tag{4}$$

satisfying the relations

$$\text{a)}\ \ h^{abc}_{\ \ \ j}h^i_{abc} = \lambda\delta^i_j, \qquad \text{b)}\ \ h^{abc}_{\ \ \ i}h^i_{dbc} = \frac{wr}{3}\delta^a_d. \tag{5}$$

For the purpose of normalizing the pseudovector ξ_α determined by the system (2), we introduce the object

$$\mathfrak{D} = \mathrm{Det}(\mathfrak{X}^\alpha, \mathfrak{X}^\alpha_a, \mathfrak{X}^\alpha_{ab}, n^\alpha_i, v^\alpha) = \mathfrak{d}_\alpha v^\alpha, \tag{6}$$

where v^α denotes the components of an arbitrary vector linearly independent of the vectors $\mathfrak{X}^\alpha, \mathfrak{X}^\alpha_a, \mathfrak{X}^\alpha_{ab}, n^\alpha_i$, and $\mathfrak{d}_\alpha$ denotes the algebraic complements of the corresponding components of the vector v^α in the determinant $\mathfrak{D}$. Under a change of the partially adjoining space H_r, the object $\mathfrak{D}$ and, consequently, the vector density $\mathfrak{d}_\alpha$ do not change. The components of the vector density $\mathfrak{d}_\alpha$ are scalar densities of weight $-(n-r)$ in E_1, of weight $m+2$ in E_m, and of weight $+1$ in E_r.

Let us now consider the W-vector density of weight -1, determined up to sign,

$$\mathfrak{t}_\alpha = |I|^{1/\lambda}\mathfrak{d}_\alpha, \tag{7}$$

whose components are scalar W-densities of weight $-n$ in E_1 and of weight $3r/m + m + 2$ in E_m. It is obvious that the density thus constructed is a solution of the system (2). The fourth-order differential form formed with the aid of the density (7),

$$\mathfrak{t}_\alpha \mathfrak{X}^\alpha_{abce}\, d\eta^a d\eta^b d\eta^c d\eta^e = A_{abce}\, d\eta^a d\eta^b d\eta^c d\eta^e \tag{8}$$

is not identically equal to zero on the surface (1); otherwise the surface under consideration lies in a hyperplane of the space P_n. At the basis of the form (8) lies the tensor density A_{abce}, whose discriminant $\mathfrak{A} = \mathrm{Dis}(A_{abce})$ is, in the general case, different from zero. (An example of a surface with nonzero discriminant $\mathfrak{A}$ is given at the end of the article.)

Using the results of works (3), we can now construct on the surface (1), from the components of the associated tensor density A_{abce}, the tensor densities $\mathfrak{A}^{abcd}$ and $\mathfrak{A}_{abce}$ of weights $+4/m$ and $-4/m$, respectively, connected by the relation $\mathfrak{A}^{abcd}\mathfrak{A}_{abce} = \delta^d_e$, and then define in the tangent stratified space $E_m(X_m)$* attached to the surface under study the object of affine connection Γ^a_{bc}. Associating with each point of the base space X_m two local spaces E_m and E_1, we obtain a doubled stratified space $E_m \times E_1(X_m)$ (²), the affine connection in which can be uniquely determined by requiring that the covariant derivative of the discriminant $\mathfrak{A}$ be equal to zero:

$$D_b\mathfrak{A} = \partial_b\mathfrak{A} + 3^{m-1}\{m(n+1)\gamma_b - [(m+1)^2 + 3(r+1)]\Gamma^c_{bc}\}\mathfrak{A} = 0. \tag{9}$$

Indeed, since the quantities Γ^a_{bc} are determined, it follows from relation (9) that the affine-connection coefficients γ_b in the radial stratified space $E_1(X_m)$ are found uniquely:

$$\gamma_b = \frac{(m+1)^2 + 3(r+1)}{m(n+1)}\,\Gamma^c_{bc} - \frac{3^{1-m}}{m(n+1)}\,\partial_b \ln\mathfrak{A}. \tag{10}$$

The basis of the invariantly adjoining space B_{n-m} is the system of the following $n-m$ vectors:

$$\bar{l}_{ab} = D_{(a}D_{b)}\bar{\mathfrak{X}}, \qquad \bar{N}_i = \frac{1}{m}\,h_i^{abc}D_aD_bD_c\bar{\mathfrak{X}}, \qquad \bar{N} = \frac{1}{m}\,\mathfrak{A}^{abce}D_aD_bD_cD_e\bar{\mathfrak{X}}. \tag{11}$$

In conclusion we give an example of a surface of special form to which the scheme of invariant adjoining set forth above is applicable. Let a three-dimensional surface in nine-dimensional projective space, with some–

* In the literature ([1],[2]) the term "compound manifold" is used instead of "stratified space."

in the second normalization the vector $\bar{x}$ has the equation

$$\begin{gathered}\bar{x} = e_0 + \eta^1\bar{e}_1 + \eta^2\bar{e}_2 + \eta^3\bar{e}_3 + \eta^1\eta^2\bar{e}_4 + \eta^1\eta^3\bar{e}_5 + \eta^2\eta^3\bar{e}_6 + \\ +((\eta^1)^2 - (\eta^3)^2)\bar{e}_7 + ((\eta^2)^2 - (\eta^3)^2)\bar{e}_8 + a_{abc}\eta^a\eta^b\eta^c e_9 + \bar{f}_4(\eta^a)\end{gathered} \tag{12}$$

$$(a, b, c = 1, 2, 3),$$

where $\bar{e}_0, \bar{e}_1, \dots, \bar{e}_9$ are vectors constituting a basis of the attached space B_{10}; $a_{abc}\eta^a\eta^b\eta^c$ is an arbitrary harmonic form (a form that is a harmonic function) of the variables η^a; $\bar{f}_4(\eta^a)$ is a vector-function of the scalar variables η^a, representing a sum of harmonic forms of order not lower than the fourth. It is obvious that in this normalization the vector $\bar{x}$ satisfies the differential equation $\bar{x}_{11} + \bar{x}_{22} + \bar{x}_{33} = 0$.

Let us consider this surface in a neighborhood of the point P_0 with curvilinear coordinates $\eta^\alpha = 0$. At the point P_0 the nine vectors $\bar{x}, \bar{x}_a, \bar{x}_{11}, \bar{x}_{12}, \bar{x}_{13}, \bar{x}_{22}, \bar{x}_{23}$ are linearly independent; by continuity they will also be linearly independent in some neighborhood of this point. The coefficients of the form (8) at the point under consideration have the form $A_{abc} = 6r_\alpha e_9^\alpha a_{abc}$ $(\alpha = 1, \dots, 10)$, where $\alpha e_9^\alpha \neq 0$; therefore the discriminant $\mathfrak{A}$ vanishes together with the discriminant of the tensor a_{abc}. The requirement that the form $a_{abc}\eta^a\eta^b\eta^c$ be harmonic, however, does not entail that the discriminant of the tensor a_{abc} be equal to zero; this can be verified by assigning the components of this tensor the values

$$a_{221} = a_{113} = 1, \quad a_{331} = a_{223} = -1, \quad a_{111} = a_{112} = a_{222} = a_{332} = a_{333} = a_{123} = 0.$$

Then, defining partially the osculating space H_5 at the point P_0 by the vectors $\bar{x}_{11}, \bar{x}_{12}, \bar{x}_{13}, \bar{x}_{22}, \bar{x}_{23}$, we obtain the following values of the essential components of the tensor density $\mathfrak{A}^{ab}$: $\mathfrak{A}^{11} = \mathfrak{A}^{22} = \mathfrak{A}^{33} = 1$, $\mathfrak{A}^{12} = \mathfrak{A}^{13} = \mathfrak{A}^{23} = 0$, and, consequently, $I = \mathrm{Dis}(\mathfrak{A}^{ab}) \neq 0$.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.