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HOMOTOPIC AND TOPOLOGICAL INVARIANCE OF SOME RATIONAL PONTRYAGIN CLASSES

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We shall consider the rational (or real) Pontryagin classes
$p_i \in H^{4i}(M^n, R)$ of closed orientable manifolds. The formula of Thom–Rokhlin –Hirzebruch is well known

$$(L_k(p_1, \dots, p_k), [M^{4k}]) = \tau(M^{4k}),$$

where L_k are the Hirzebruch polynomials and τ is the index of the manifold. In questions of invariance of the classes, only the following was known:

1. Homotopy invariance–only the formula $(L_k, [M^{4k}]) = \tau(M^{4k})$; for simply connected manifolds there exist no further invariance relations (a discussion of this may be found in ([1]), Appendix 1).
2. Combinatorial invariance–all classes are invariant (see ([3,4])).
3. Topological invariance, as Rokhlin showed in 1956 ([2]): the class $L_k(M^{4k+1})$ is invariant.

It will be seen below (Theorem 1) that the class $L_k(M^{4k+1})$ is even a homotopy invariant; therefore Rokhlin' s result on $L_k(M^{4k+1})$ did not yield relations of topological invariance of classes distinct from homotopy invariance.

As was clarified by Thom and Rokhlin ([2,4]), in questions of invariance it is useful to pass from the classes p_k to the classes L_k.

The results of this note are as follows.

Theorem 1. The class $L_k(M^{4k+1})$ is a homotopy invariant.

Theorem 2. Let $n = 4k + 2$, $x \in H_{4k}(M^n, Z)$ be an indivisible element such that
$Dx = y_1 \wedge y_2$, where $y_1, y_2 \in H^1(M^n, Z)$, and D is the Poincaré duality operator. Then the scalar product (L_k, x) is homotopy invariant if the element x has the following properties: consider a covering $M \to M^n$ under which precisely those

and only those paths γ in M^n become closed for which $(\gamma, y_1) = (\gamma, y_2) = 0$; it is required that the homology group $H_{2k+1}(\widetilde{M}, R)$ be finite-dimensional.

The next theorem already concerns essentially topological invariance.

Theorem 3. Let $n = 4k+2$, $x \in H_{4k}(M^n, Z)$ be such an element that $(Dx)^2 = 0$. Then the scalar product (L_k, x) is topologically invariant.

Corollary 1. It follows from Theorem 2 that the class $L_k(T^{4k+2})$ of the torus T^n (or $p_1(T^6)$) is homotopy invariant; moreover, the class $L_k(M^{4k} \times T^2)$ has an invariant scalar product with the cycle $M^{4k} \times O$.

Corollary 2. It follows from Theorem 3 that the Pontryagin class $p_k(S^2 \times S^{4k})$ is topologically invariant. The class $L_k(M^{4k+2})$ for $M^{4k+2} = M^{4k} \times S^2$ has an invariant scalar product with the cycle $M^{4k} \times O$.

Corollary 3. Hurewicz' s problem on the distinction between homotopy type and homeomorphism of closed manifolds was solved by Milnor only for $n = 3$, on the basis of the "Hauptvermutung," in the 1950s.

Moreover, the examples for $n = 3$ (lens spaces) are not simply connected and have different "simple" homotopy type. For $n > 3$ the problem remained open.

Since the J-functor of Atiyah $\mathcal{J}_R(X)$ cited above is always finite, we can apply the results of the author and of Browder on normal bundles (see (1), §14) to $X = S^2 \times S^{4k}$ and obtain already for $S^2 \times S^4$ an infinite number of homotopy-equivalent simply connected manifolds with different Pontryagin classes (homotopy type $S^2 \times S^4$).

Theorem 4. *In every dimension of the form $4k+2$, $k \geqslant 1$, there exists an infinite number of non-homeomorphic manifolds of the homotopy type $S^2 \times S^{4k}$. For $k \geqslant 2$, among these manifolds there are PL-manifolds, not homeomorphic to smooth ones and with fractional Pontryagin class $p_k(M^{4k+2})$.*

The last part of the theorem follows from the combinatorial J-functor and the results of (1), Appendix 2, which generalize the theorems of Browder and of the author to the combinatorial case (J-functor and normal bundles). For $k = 2$ the denominator of the class $p_2(M^{10})$ may be taken equal to 7. In the previously known examples, nonsmoothability was of a cruder—homotopy—character.

We give an outline of the proof, first of Theorems 1 and 2.

I. Consider a covering $\widehat{M} \to M^{4k+1}$ under which precisely the paths γ having intersection index zero with the basic cycle $x \in H_{4k}(M^{4k+1})$ are covered as closed paths. (The analogous covering for M^{4k+2} has already been described in the formulation of Theorem 2.) On $\widehat{M}$ acts the group Z, generated by the transformation $T : \widehat{M} \to \widehat{M}$. In the case $n = 4k+2$, on $\widehat{M}$ acts $Z+Z$, generated by $T_1 : \widehat{M} \to \widehat{M}$ and $T_2 : \widehat{M} \to \widehat{M}$.

Lemma 1. *There is a cycle $\hat{x} \in H_{4k}(\widehat{M})$ such that $T_* \hat{x} = \hat{x}$ for $n = 4k+1$, and $T_{1*}\hat{x} = T_{2*}\hat{x} = \hat{x}$ for $n = 4k+2$, and such that $p_* \hat{x} = x$.*

This element $\hat{x}$ is uniquely determined if one requires that $\hat{x} \otimes [T^l]$, $l = 1, 2$, be a cycle of all differentials of the spectral sequence of the fibration $M^n \to T^l$, dual to the covering. We shall call it canonical.

For any complex X and cycle $z \in H_{4k}(X)$ one can introduce an invariant $\tau(z)$: consider the quadratic form (y^2, z) on $H^{2k}(X, R)$ and discard its "degenerate part"; a finite-dimensional nondegenerate quadratic form remains, and one must take its index—this is $\tau(z)$.

We have already chosen the element $\hat{x} \in H_{4k}(\widehat{M})$ in Lemma 1. It is determined by the element $x \in H_{4k}(M^n)$.

There is a fundamental formula computing the class $L_k(p_1, \dots, p_k)$ through homotopy invariants of the manifold M^n $(n = 4k+1, 4k+2)$ under the hypotheses of Theorems 1 and 2:

$$(L_k(p_1, \dots, p_k), x) = \tau(\hat{x}),$$

where $\hat{x}$ is the canonical cycle on $\widehat{M}$. This formula may be interpreted as the correct generalization of the Hirzebruch formula $(L_k, [M^{4k}]) = \tau(M^{4k})$, if one regards M^{4k} itself as a "trivial covering" of itself and puts $x = |x| = [M^{4k}]$. From the formula it is clear that, to compute the Pontryagin class, one must take the covering as many times as the codimension of the cycle being studied.

We indicate one trivial property of the form and of the index $\tau(z)$.

Lemma 2. *Let $X_1 \subset \dots \subset X_k \subset \dots$ be an embedding of finite complexes, $X = \bigcup X_i$, $j_k : X_1 \subset X_k$. Let $z \in H_{4k}(X_1)$ be an element such that $j_* z \neq 0$ in X. Then, for sufficiently large indices k, we have $\tau(j_{k*} z) = \tau(j_* z)$. Moreover, the nondegenerate part of the quadratic form $[y^2, j_{k*} z]$ stabilizes for sufficiently large indices k.*

II. Consider $n = 4k+1$ and realize the cycle $x \in H_{4k}(M^n)$ by a submanifold. All paths on the submanifold $M^{4k} \subset M^n$ are covered in $\widehat{M}$ by closed—

pulled tight; hence the embedding $j : M^{4k} \subset \hat{M}$ is defined and $TM^{4k} \cap M^{4k} = \varphi$; moreover, between TM^{4k} and M^{4k} there lies a film N, $\partial N = M^{4k} \cup TM^{4k}$. Clearly, $\hat{M} = \bigcup_i T^{iN}$. Put

$$M_1 = \bigcup_{i \geq 0} T^{iN}, \qquad M_2 = \bigcup_{i<0} T^{iN}, \qquad M_1 \cup M_2 = \hat{M}, \qquad M_1 \cap M_2 = M^{4k}.$$

The embeddings $M^{4k} \subset M_1$ and $M^{4k} \subset M_2$ will be denoted by j_1 and j_2. Put $J = \operatorname{Im} j^*$, $J_1 = \operatorname{Im} j_1^*$, $J_2 = \operatorname{Im} j_2^* \subset H^{2k}(M^{4k}, R)$. The form $(y^2, [M^{4k}])$ induces forms on J, J_1, and J_2, whose indices we denote by $\tau(J)$, $\tau(J_1)$, $\tau(J_2)$. Clearly, $J = J_1 \cap J_2$. Trivially we have

$$\tau(J) = \tau(j_*[M^{4k}]) = \tau(\hat{x}),$$
$$\tau(J_1) = \tau(j_{1*}[M^{4k}]) = \tau(\hat{x}_1),$$
$$\tau(J_2) = \tau(j_{2*}[M^{4k}]) = \tau(\hat{x}_2),$$
$$\hat{x} = j_*[M^{4k}], \qquad \hat{x}_1 = j_{1*}[M^{4k}], \qquad \hat{x}_2 = j_{2*}[M^{4k}].$$

Lemma 3. $\tau(\hat{x}) = \tau(\hat{x}_1) = \tau(\hat{x}_2) = \tau(J) = \tau(J_1 \cup J_2)$.

The proof follows from Lemma 2 and from the observation that the transformation T^l makes it possible to bring arbitrarily large parts of M_1, M_2, and M into coincidence. The conclusion about the index $\tau(J_1 \cup J_2)$ is made on the basis of simple algebra of quadratic forms, since the whole nondegenerate part J_i is concentrated on $J = J_1 \cap J_2,\ i = 1, 2$.

Lemma 4. $\tau(J_1 \cup J_2) = \tau(M^{4k})$.

If $\alpha \in H^{2k}(M^{4k}, R)$, and $(\alpha J_1, [M^{4k}]) = (\alpha J_2, [M^{4k}]) = 0$, then the element $\beta = \alpha \cap [M^{4k}]$ is such that $j_{1*}\beta = j_{2*}\beta = 0$. The films on the right and on the left, stretched over β, define a cycle $\delta \in H_{2k+1}(\hat{M}, R)$. A compact cocycle $D_k\delta$ in $\hat{M}$ is such that $j^* D_k \delta = \alpha$. It follows that $\alpha \in J$. (The embedding of compact cohomology in ordinary cohomology is a permutation with j^*.) Lemma 4 follows easily from this.

Remark. If δ is homologous to zero, then the argument with duality D_k is unnecessary—then $\alpha \cap [M^{4k}] = 0$. Moreover, it suffices that δ be homologous to zero modulo infinity by means of a film having compact intersection with M^{4k}. In further applications arguments of this kind will be very substantial.

Theorem 1 now follows easily from the lemmas. Indeed,

$$(L_k, x) = (L_k, p_*\hat{x}) = (p^* L_k, \hat{x}) = \tau(\hat{x}).$$

III. We pass to Theorem 2. Realize the cycles Dy_1, Dy_2 by the manifolds M_1^{4k+1}, $M_2^{4k+1} \subset M^{4k+2}$, and the cycle x by the intersection $M_1^{4k+1} \cap M_2^{4k+1} = M^{4k}$. Then we have: a) over M^{4k} the covering is trivial; b) over M_i^{4k+1} the covering splits into an infinite number of coverings with group Z. Put

$$p^{-1}(M_1^{4k+1}) = \bigcup_q M_q, \qquad \text{where } T_2 M_q = M_{q+1}, \quad T_1 M_q = M_q.$$

On M_q there is a cycle $t_q \in H_{4k}(M_q)$ such that $j_{q*} t_q = \hat{x}$, where $j_q : M_q \subset \hat{M}$, and $T_{1*} t_q = t_q$. From the preceding we conclude that $\tau(t_q) = \tau(M^{4k})$, since M^{4k} realizes t_q on M_q, and the film between t_q and $T_1 t_q$ lies between M_q and $T_1 M_q$.

By an analogous argument we can show that $\tau(t_q) = \tau(\hat{x})$. At this second step, however, we use the finite-dimensionality of $H_{2k+1}(\hat{M}, R)$ in order to carry out an argument of the type of Lemma 4 (this is indicated in the remark to Lemma 4). From finite-dimensionality it follows that for any cycle

$$\delta \in H_{2k+1}(\hat{M}, R)$$

there is a relation of the form

$$\delta = \sum_{s=1}^{N(\delta)} \lambda_s T_{2*}^s \delta,$$

whence we can con–

···conclude that the cycle δ is homologous to zero on $\hat{M}$ modulo infinity in such a way that the film has compact intersection with $M_q \subset \hat{M}$. The rest is analogous to the preceding.

Theorem 2 also now follows from the lemmas.

IV. We pass to the topological part. In one smoothness structure on M^{4k+2} we realize the cycle x by a submanifold with trivial normal bundle $M^{4k} \times R^2 \subset M^{4k+2}$, which can be done by virtue of the condition $(Dx)^2 = 0$. Consider the natural embedding $M^{4k} \times S^1 \times R \subset M^{4k} \times R^2$, and in any other smoothness structure we shall realize this same cycle in $M^{4k} \times S^1 \times R$. Let another smoothness structure be given. Realize the cycle $M^{4k} \times S^1$ by a smooth $W^{4k+1} \subset M^{4k} \times S^1 \times R$. A projection f of degree $+1$ is defined: $W^{4k+1} \to M^{4k} \times S^1$. Consider the covering over $M^{4k} \times S^1 \times R$ which preserves as closed all paths on $M^{4k} \times O$. It is easy to see that $\hat{M}$ is homeomorphic to $M^{4k} \times R \times R$, and $\hat{N} = p^{-1}(W^{4k+1})$ is a covering over W^{4k+1} with group Z, generated by the transformation $T : N \to N$. There is a cycle $z \in H_{4k}(N)$ such that $T_* z = z$, and $p_* z$ is homologous to $M^{4k} \times O$; moreover, z is homologous to $M^{4k} \times O \times O$ in $\hat{M}$. I assert that $\tau(z) = \tau(M^{4k})$ and $\tau(z) = (L_k(W^{4k+1}), p_* z)$. The second part has already been proved above. For the first equality we consider the transformation $T' : M^{4k} \times R \times R \to M^{4k} \times R \times R$, where $T'(m, s, t) = (m, s, t+1)$, $m \in M^{4k}$, and we assume that W^{4k+1} lies between the levels $t = 0$ and $t = 1$. The transformation T' may not be smooth. Denote the embedding $N = p^{-1}(W^{4k+1}) \subset \hat{M}$ by j. Then $T_*(j_* z) = j_* z$, and the group $H_{2k+1}(\hat{M}) = H_{2k+1}(M^{4k})$ is invariant under T'_*. Analogously to the preceding, $\tau(z) = \tau(j_* z) = \tau(M^{4k})$. This implies the assertion of the theorem, since the normal bundle of W^{4k+1} in $M^{4k} \times S^1 \times R$ (in the smoothness structure under consideration) is trivial, and the class $L_k(W^{4k+1})$ is cut out by the class $L_k(M^{4k} \times S^1 \times R)$, and hence by the class $L_k(M^{4k+2})$. The proof of the theorem is complete.

Remark added in proof. At the present time the author has completely proved the topological invariance of all rational Pontryagin classes. A brief account will appear in (5).

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REFERENCES

[1] S. P. Novikov, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **28**, No. 2, 365 (1964).
[2] V. A. Rokhlin, *DAN*, **113**, No. 2, 276 (1957).
[3] V. A. Rokhlin, A. S. Schwarz, *DAN*, **114**, No. 3, 490 (1957).
[4] R. Thom, *Symp. Int. Top. Alg.* (Mexico), 1958.
[5] S. P. Novikov, *DAN*, **163**, No. 2 (1965).

Note: Figure translations are in progress. See original paper for figures.