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P. G. AIZENGENDLER, M. M. VAINBERG

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Abstract

Full Text

MATHEMATICS

P. G. AIZENGENDLER, M. M. VAINBERG

THEORY OF BRANCHING OF SOLUTIONS OF NONLINEAR EQUATIONS IN THE MULTIDIMENSIONAL CASE

(Presented by Academician G. I. Petrov, 19 III 1965)

1. Let $F(x, \lambda)$ be an analytic operator acting from some neighborhood of the point $(0, 0)$ of the topological product $E \times \Lambda$ into E_1 , where E and E_1 are Banach spaces and Λ is the complex plane, with $F(0, 0) = 0$. We shall assume that the equation $F(x, \lambda) = 0$ can be written in the form ⁽³⁾

$$Bx = F_{01}\lambda + \sum_{k+s \geq 2} C_{k+s}^s F_{ks} x^k \lambda^s, \tag{1}$$

where B is a linear operator from E into E_1 with closed range; F_{ks} are homogeneous operators of order k in x and order s in λ , and the null spaces of the operators B and B^* have the same finite dimension r , and the series (1) converges in some neighborhood of the point $(0, 0)$.

For $r = 0$, equation (1) has a unique small solution* $x = x(\lambda)$. For $r > 0$, at the point $\lambda = 0$ branching of solutions of equation (1) is possible, i.e., in some neighborhood of the point $\lambda = 0$ the number of small solutions may differ from one.

Using the Lyapunov-Schmidt method, we reduce equation (1) to the system

$$x = \sum_{m_1 + \dots + m_r + m \geq 1} X_{m_1, \dots, m_r, m} \xi_1^{m_1} \dots \xi_r^{m_r} \lambda^m, \tag{2}$$

$$\begin{aligned} \Phi_i(\xi_1, \dots, \xi_r, \lambda) \equiv & \sum_{m_1 + \dots + m_r \geq 2} L_{m_1, \dots, m_r, 0}^{(i)} \xi_1^{m_1} \dots \xi_r^{m_r} + \\ & + \sum_{m_1 + \dots + m_r > 0} \xi_1^{m_1} \dots \xi_r^{m_r} \sum_{m \geq 1} L_{m_1, \dots, m_r, m}^{(i)} \lambda^m = 0, \quad i = 1, \dots, r, \end{aligned} \tag{3}$$

and show that the coefficients $X_{m_1, \dots, m_r, m} \in E$ are determined uniquely by recurrence formulas, and the series (2) and (3) converge in some neighborhood of the point $\xi_1 = \dots = \xi_r = \lambda = 0$. For computing the coefficients $L^{(i)}$ of

the system of branching equations (3), convenient formulas are obtained. Thus, each small solution of system (3) gives, by formula (2), one small solution of equation (1).

For $r = 1$, system (3) becomes a single equation, and if it is nondegenerate, i.e., not all of its coefficients are zero, then the number of its small solutions is finite and each such solution is representable in the form of a convergent series in integral or fractional powers of λ . The number of these solutions and the form of each are easily determined with the aid of the Newton diagram⁽³⁾. In the degenerate case (when all coefficients of the branching equation are equal to zero), as is seen from (2), equation (1) has a family of small solutions depending on an arbitrary parameter ξ_1 . For $r > 1$, only individual results are known.

* A solution $x(\lambda)$ is called small if $\lim_{\lambda \rightarrow 0} x(\lambda) = 0$, and $x(\lambda)$ is defined in an open set $\omega \subset \Lambda$ whose boundary $\omega' \ni 0$.

In the present paper we give a complete solution of the problem of the branching of small solutions of equation (1) for $r > 1$. Namely, we distinguish two cases—the nondegenerate case (when the number of small solutions of equation (1) is finite and each such solution can be represented in the form of a convergent series in integral or fractional powers of λ) and the degenerate case (when equation (1) has a family of small solutions depending on one or more arbitrary parameters ξ_i); then we prove that these exhaust all possible cases.

2. Let $K[z]$ be a domain of polynomials, where K is a domain with unique factorization. As is known⁽⁵⁾, $K[z]$ is also a domain with unique factorization; consequently, for any two polynomials $f(z)$ and $g(z)$ from $K[z]$ there exists a greatest common divisor (g.c.d.).

Following Osgood, we give the following definition. A primitive common divisor that is divisible by any primitive common divisor of the polynomials $f(z)$ and $g(z)$ will be called the **primitive g.c.d.** for $f(z)$ and $g(z)$. Let the polynomials $f(z), g(z) \in K[z]$; $d(z)$ be their primitive g.c.d.; a_0 the leading coefficient; R_σ the first nonzero subresultant of the polynomials $f(z)$ and $g(z)$; and $D(z)$ the polynomial obtained by replacing the last column of the determinant R_σ by the expressions $\dots z^2 f(z), z f(z), f(z), g(z), z g(z), \dots$ (see⁽¹⁾, p. 184).

Lemma 1. The degree of $d(z)$ is equal to σ , and the relation

$$a_0 D(z) = R_\sigma d(z). \quad (4)$$

holds.

Lemma 2. Let a_0 and b_0 be the leading coefficients of the polynomials $f(z)$ and $g(z)$, respectively. Then, if a_0 and b_0 are associated with unity ($a_0 \sim b_0 \sim 1$), the primitive g.c.d. coincides with the g.c.d., and (4) takes the form

$$D(z) = R_\sigma d(z). \quad (4_1)$$

We apply all these concepts to the investigation of the system of branching equations (3).

3. Let $K_n = K[[t_1, \dots, t_n]]$ be the set of power series (with complex coefficients) in n complex variables t_1, t_2, \dots, t_n , each of which converges in some (its own) neighborhood of the point $t_1 = t_2 = \dots = t_n = 0$. It is known ⁽⁶⁾ that K_n is a domain with unique factorization. Then for the polynomial ring $K_n[z]$ the assertions of item 2 are valid. In particular, for the distinguished polynomials ⁽²⁾ from $K_n[z]$ the conditions of Lemma 2 are fulfilled, and the g.c.d. of two such polynomials is also a distinguished polynomial.

4. Since we are interested in small solutions of the system (3), applying first a nonsingular linear transformation of the variables (ξ_1, \dots, ξ_r) , and then the Weierstrass preparation theorem, we reduce the system (3) to the system

$$G_i^{(1)}(\xi_1^{(1)}, \xi_2^{(1)}, \dots, \xi_r^{(1)}, \lambda) = 0, \quad i = 1, 2, \dots, r, \quad (3_1)$$

where $G_i^{(1)}$ are distinguished polynomials with respect to $\xi_1^{(1)}$ of degree s_i (s_i is the order of the series $\Phi_i(\xi_1, \dots, \xi_r, 0)$). Each small solution of the system (3) gives a small solution of the system (3₁), and conversely. To find small solutions of the system (3₁) we shall use Kronecker's method of elimination ⁽⁴⁾. Let d_1 be the g.c.d. of the system of polynomials $G_i^{(1)}$, so that $G_i^{(1)} = d_1 g_i^{(1)}$. By Kronecker's method, from the system $g_i^{(1)} = 0$ we eliminate $\xi_1^{(1)}$. We obtain the system

$$\Phi_j^{(2)}(\xi_2^{(1)}, \xi_3^{(1)}, \dots, \xi_r^{(1)}, \lambda) = 0, \quad j = 1, 2, \dots, r_2, \quad (5)$$

in which it may be assumed that $\Phi_j^{(2)}(\xi_2^{(1)}, \dots, \xi_r^{(1)}, 0) \neq 0$. Applying again to the system (5) a nonsingular linear transformation and the Weierstrass preparation theorem, we reduce the system to the form

$$G_j^{(2)}(\xi_2^{(2)}, \xi_3^{(2)}, \dots, \xi_r^{(2)}, \lambda) = 0, \quad j = 1, 2, \dots, r_2, \quad (3_2)$$

where $G_j^{(2)}$ are the indicated polynomials with respect to $\xi_2^{(2)}$. Again we isolate the g.c.d. (denote it by d_2) and continue the process. In this way we find d_1, d_2, \dots, d_k , where $k \leq r$. The following basic theorem holds.

Theorem 1. *In order for the nondegenerate case to occur, it is necessary and sufficient that the following conditions be fulfilled: 1) $k = r$; 2) $d_i \sim 1$, $i = 1, \dots, r - 1$. Moreover, if at least one of these conditions is violated, then the degenerate case occurs.*

Let us note that if, in the nondegenerate case, the polynomial d_r is not associated with unity, then the number of small solutions is greater than zero.

The method proposed here makes it possible to investigate the degenerate case in greater detail as well. In particular, if d_r is not associated with 1, then one can isolate a finite number of small solutions of equation (1), represented in the form of series in integral or fractional powers of the parameter λ , which, possibly, are not obtained from the family of solutions of equation (1) depending on a certain number of arbitrary parameters ξ_i .

5. For actually finding the small solutions of equation (1), it is enough to find the small solutions of system (3) and substitute them into formula (2). One may, however, seek the small solutions of equation (1) in the form of formal series in integral and fractional powers of the parameter λ .

Theorem 2. *In the nondegenerate case every formal solution is also a genuine one (i.e., a convergent one).*

In the degenerate case there are also possible formal solutions of equation (1) which converge only for $\lambda = 0$. For example, for the equation

$$x(t) = \frac{1}{\pi} \int_0^{2\pi} \cos(t-s)x(s) ds + \lambda \int_0^{2\pi} (\sin t \sin s)x^2(s) ds,$$

as the calculation shows, $r = 2$, and the case is degenerate; for it the function

$$x(t) = \sum_{n=1}^{\infty} n! \lambda^n (\sin t + \cos t)$$

is a formal solution.

Theorem 2 and this example show that only in the degenerate case is it expedient to investigate the question of convergence of formal solutions.

6. We assumed that the equation $F(x, \lambda) = 0$ can be written in the form (1), and showed that only two basic cases are possible—the degenerate and the nondegenerate. Investigation of the question of the relation between these two cases and the isolatedness of the zero solution of the equation $F(x, 0) = 0$ leads to the proposition:

Theorem 3. *If the zero solution of the equation $F(x, 0) = 0$ is isolated, then for equation (1) the nondegenerate case occurs.*

The converse assertion does not always hold. For example, for the equation

$$x(t) = \frac{2}{\pi} \int_0^{\pi} \cos(t-s)x(s) ds + 3\lambda \int_0^{\pi} \sin(t-s)x^2(s) ds + \lambda^2(\sin t - \cos t) \quad (6)$$

the nondegenerate case occurs. However, for $\lambda = 0$ the zero solution of equation (6) is not isolated.

7. The formulation of Theorem 1 uses expressions obtained by the Kronecker elimination method. It is known, however, that Kronecker's method leads to cumbersome computations. In this connection, we have

* An analogous proposition is contained in (7), but there the proof essentially uses additional restrictions.

another method is considered for eliminating the unknowns of system (3), consisting in a pairwise compilation of the results, which in our view is more economical. It leads to another formulation of what we have in the nondegenerate case.

Moscow Regional Pedagogical Institute
named after N. K. Krupskaya

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