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Abstract

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ON THE PROBLEM OF DIVISORS IN SEGMENTS OF ARITHMETIC PROGRESSIONS

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§ 1. The problem of divisors in segments of arithmetic progressions, which has numerous and important applications, consists in deriving an asymptotic formula for sums of the form

$$\sum_{dn+l\le x} \tau_k(dn+l), \tag{1}$$

with $d \le x^{\gamma}$, $0 \le l \le d$ for as large γ as possible, where $\tau_k(m)$ denotes the number of all decompositions of m into k natural factors.

For $k = 4$, $d \le \sqrt[5]{x}$, with $\sqrt[5]{x}$ close to $\sqrt{x}$, this problem was solved by Yu. V. Linnik in (^{1}) on the basis of the method of the shortened functional equation for Dirichlet L-functions.

In the present paper results are given which generalize and further develop the ideas of this work of Yu. V. Linnik.

For sums (1) the following holds.

Theorem 1. For integers $k \ge 4$, $4 \le 2m \le k$, $(l, D) = 1$, uniformly in $D > 1$, $0 \le l < D$,

$$\sum_{\substack{n\le x \\ n\equiv l \pmod D}} \tau_k(n) = \frac{x}{\varphi(D)} \sum_{d \mid D^k} \sum_{d=d_1\dots d_k} \frac{\mu(d_1)\dots\mu(d_k)}{d} P_k\Big(\ln\frac{x}{d}\Big) + R, \tag{2}$$

where

$$R = R_{k,m}(x, D) \ll \frac{1}{\varphi(D)} x^{1-1/2^{\nu}m} D^{(3k+2m)/2^{\nu}+3m+\varepsilon_0} \ln^b x;$$

φ is Euler’ s function; μ is the Möbius function; P_k is a polynomial of degree $k-1$, determined by the equality

$$P_k\Big(\ln\frac{x}{d}\Big) = \frac{d}{(k-1)!\,x} \lim_{s\to 1} \frac{d^{k-1}}{ds^{k-1}} \left\{ \frac{\zeta^k(s)(s-1)^k}{s} \left(\frac{x}{d}\right)^s \right\};$$

ζ is the Riemann zeta-function; ν is an integer $\geq (2k-3m)/m$; $\varepsilon = 0$ or an arbitrarily small $\varepsilon > 0$, according as $2m = k$ or $2m < k$; b is a constant depending only on k.

Since the main term in (2) is of order $x \ln^{k-1} x/\varphi(D)$, (2) gives for the asymptotics of sums (1) the range

$$D \ll \frac{x^{\gamma}}{\ln^{b} x}, \qquad \gamma = \frac{8}{3k+2m} - \varepsilon_0,$$

and, correspondingly, a reduction in the remainder term by

$$x^{-\beta}, \qquad \beta = \frac{1}{2^{\nu} m}\left[1 - \gamma\left(\frac{3k+2m}{8} + \varepsilon\right)\right].$$

It is seen from this that the best range for D is attained when $m = 2$, with $\gamma = 8/(3k+4) - \varepsilon_0$, and that in this case the reduction of the remainder with increasing k decre–

decreases according to a power law ($\nu = k-3$). In the other extreme case $2m = k$ we have a bound for D with $\gamma = 2/k$, but the reduction of the remainder with increasing k decreases only as $x^{-\beta}$, with $\beta = 1/k - \gamma/2$.

Let us note that the case of primitive progressions, considered in Theorem 1, is the basic one, since any other case is reduced to it in an entirely elementary way (^{1,2}).

§ 2. By means of a new truncated equation, Yu. V. Linnik (^{1}) achieved a fundamental advance in estimates of $L(1/2+it,\chi)$ on the real axis and near it ($|t| <$ const).

A further development of the idea of this equation is the following

Theorem 2. *If χ is a primitive character, $d > 1$, $z \neq 0$, $|\arg z| \leq \pi/2$, then in the whole plane of the variable s, except for the points $s = 0, -1, -2, \ldots$, the equation holds*

$$\Gamma\left(\frac{s+a}{2}\right) L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \Gamma\left(\frac{s+a}{2}, \frac{\pi n^2 z}{d}\right) +$$

$$+\varepsilon(\chi)\left(\frac{d}{\pi}\right)^{1/2-s} \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n^{1-s}} \Gamma\left(\frac{1-s+a}{2}, \frac{\pi n^2}{dz}\right), \tag{3}$$

where $\varepsilon(\chi)$ and a denote the same as in the usual equation for $L(s,\chi)$;

$$\Gamma(a,u) = \int_u^{\infty} e^{-x} x^{a-1}\, dx; \qquad \Gamma(a) \text{ is the gamma function.}$$

The meaning of the new form (3) of the functional equation for $L(s,\chi)$ is that this equation is, in essence, truncated and makes it possible to obtain estimates of these functions in the critical strip, uniformly in d and s. Namely, if one takes

$$z = \exp i \operatorname{sign} \operatorname{Im} s \cdot \left(\frac{\pi}{2} - \frac{1}{|\operatorname{Im} s| + 1} \right),$$

then the left-hand side of (3) will be determined (up to an accuracy $O(1)$) by segments of the series on the right-hand side with $n \le N$, where

$$N = \sqrt{d(|t|+1) \ln d(|t|+1)},$$

and in this process from each term there will be separated a factor canceling the growth of the function $\Gamma^{-1}((s+a)/2)$.

More precisely, the following is true.

Theorem 3. *If χ is a primitive character modulo $d > 1$, $s = \sigma + it$, $0 < \sigma < 1$, t real, then*

$$L(s,\chi) = \sum_{n \le N} \frac{\chi(n)}{n^s} c_n(s,d,a) +$$

$$+ \varepsilon(\chi) \left(\frac{d}{\pi} \right)^{1/2-s} \frac{\Gamma((1-s+a)/2)}{\Gamma((s+a)/2)} \sum_{n \le N} \frac{\overline{\chi}(n)}{n^{1-s}} c_n(1-s,d,a) + O(N^{-b}),$$

where $c_n(\omega,d,a)$ is a quantity bounded in all its arguments by an absolute constant; b is an arbitrary positive constant.

From Theorem 2 there also follows

Theorem 4. *Let m be an integer,*

$$N_m = \sqrt{\frac{d|t|}{2\pi}} + m \sqrt{\frac{d}{2\pi}}, \qquad N = d(|t|+1), \qquad T = \sqrt{|t|+1} \ln N.$$

Then, for a primitive character χ modulo $d > 1$, $s = \sigma + it$, $0 < \sigma < 1$, and arbitrary real t,

$$L(s,\chi) = A \sum_{|m| \le T} \left\{ \max_{v \le N_m} \left| \sum_{\substack{N_{m-1} < n \le v \\ n > 0}} \frac{\chi(n)}{n^s} \right| + \right.$$

$$\left. + B N^{1/2-\sigma} \max_{u \le N_m} \left| \sum_{\substack{N_{m-1} < n \le u \\ n > 0}} \frac{\overline{\chi}(n)}{n^{1-s}} \right| \right\} + O(N^{-b}),$$

where A, B are quantities bounded by absolute constants, and b is an arbitrary positive constant.

The latter equation has an especially simple form when $\sigma = 1/2$. Namely, if $s = 1/2 + it$, then

$$L(s,\chi) = A \sum_{|m|\le T} \max_{v\le Nm} \left| \sum_{\substack{N_{m-1}<n\le v\\ n>0}} \frac{\chi(n)}{n^s} \right| + O(N^{-b}).$$

Thus, instead of a complete sum over $1 \le n \le \sqrt{d}\,|t| \ln d|t|$, we have $T = \sqrt{|t|}\ln d|t|$ separate, nonoverlapping intervals of length $< \sqrt{d}$.

The derivation of Theorem 2 rests on the usual functional equation for $L(s,\chi)$. Let us note that, in view of the symmetry of the right-hand side of (3), this equation, in turn, follows in an obvious way from (3). As special cases, (3) contains the equation of P. O. Kuz' min [3] ($z = i$) and its generalization from [4] ($z = \delta i$), as well as the equations indicated in [1,2,5]; moreover, the case $z = \delta i$, $\delta > 0$, is apparently not the best one, because for such values of z the series on the right-hand side of (3) cease to be absolutely convergent.

The derivation of Theorem 1 is carried out on the basis of the approximate equation of Theorem 3 with the aid of D. A. Burgess' s estimate [6] for character sums. Less precise results were indicated earlier [2,7]*.

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* In paper [7] there are many inaccuracies.

Note: Figure translations are in progress. See original paper for figures.

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