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**Abstract**

**Full Text**

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## INFINITESIMAL BENDINGS OF SURFACES UNDER SLEEVE CONSTRAINTS

*(Presented by Academician I. N. Vekua, November 9, 1964)*

**1.** An external constraint of a surface is called **rigid** if the surface admits no infinitesimal bendings compatible with this constraint; otherwise the constraint is called **nonrigid**.

In the present work we shall consider infinitesimal bendings of convex surfaces with plane boundary under the condition that the external constraint is a sleeve constraint. By a sleeve constraint one means the following (see, for example, the book of I. N. Vekua <sup>(1)</sup>). Consider a simply connected surface  $S$  with smooth boundary  $L$ . Let it lie on a certain surface  $\Sigma$ . Subject the surfaces  $S$  and  $\Sigma$  to such an infinitesimal bending that, during the process of bending, contact between them is not broken; in other words, so that under the bending of the surfaces  $S$  and  $\Sigma$  the contour  $L$  moves along the deformed surface  $\Sigma$ , sliding over it. In this case the surface  $\Sigma$  is called a **surface of sliding**, or a **sleeve**, and the constraints imposed in this way on the contour  $L$  of the surface  $S$  are called sleeve constraints. A sleeve constraint is defined analogously for a doubly connected surface  $S$ . In this case the sleeve is the collection of two surfaces  $\Sigma_1$  and  $\Sigma_2$  on which the boundary of the surface  $S$  lies. We shall call a sleeve constraint **elastic** if, under a bending of  $S$ , the sleeve  $\Sigma$  behaves as a rigid body. If, under infinitesimal bendings of the surface  $S$ , infinitesimal bendings of the sleeve are also allowed, then we shall call the sleeve constraint **nonelastic**.

The main purpose of the work is to establish that, under certain conditions to be indicated below, an elastic sleeve constraint is rigid, whereas a nonelastic sleeve constraint, as a rule, always turns out to be nonrigid. The main results of the work are contained in Theorems 1-4.

**2.** In the work, infinitesimal bendings of the following classes of surfaces are considered:

- a) caps\*, given by the equation  $Z = f(x, y)$ , in a neighborhood of the vertex  $(x_0, y_0)$  of which

$$Z = (x - x_0)^2 + (y - y_0)^2.$$

Here we assume that the domain  $D$  of variation of  $(x, y)$  admits a conformal mapping  $w = \chi(z)$  onto a certain disk, satisfying the conditions

$$\left| \operatorname{Im} \left( \frac{\partial}{\partial z} \left( \frac{\chi(z)}{\chi'_z(z)} \right) \right) \right| < \varepsilon, \quad (1)$$

where  $\varepsilon$  is a sufficiently small number,  $\varepsilon > 0$ ;

$$\chi(z_0) = 0,$$

where  $z_0 = x_0 + iy_0$  is the image of the vertex of the surface in the plane  $z = x + iy$ ;

- b) belts, which are single-valuedly projected onto the plane of the boundary into a domain  $D$  admitting a conformal mapping  $w = \chi(z)$  onto an annulus under condition (1).

Application of the Darboux-Sauer theorem (see, for example, (1)) on the connection between projective transformations and infinitesimal bendings permits

\* A **cap** is a surface with plane boundary that is single-valuedly projected onto the plane of the boundary; a **vertex of the surface** is a point at which the tangent plane is parallel to the plane of the boundary. A **belt** is a doubly connected surface with plane boundary lying in parallel planes.

extend the results obtained to the following surfaces, star-shaped with respect to the center:

- a) simply connected surfaces whose boundary is a circle, and whose vertex is projected to the center of the circle;
- b) doubly connected surfaces—belts whose boundary consists of concentric circles.

Moreover, in the paper it is assumed that all the surfaces under consideration have positive Gaussian curvature up to and including the boundary and belong to the class  $C_\alpha^4$ ,  $0 < \alpha < 1$  (the radius vector  $\mathbf{r}(u, v)$  of the surface is four times continuously differentiable, and the fourth derivative satisfies the Hölder condition with exponent  $\alpha$ ).

3. Let  $S$  be a convex surface of positive curvature  $K \geq k_0 > 0$ ,  $S \in C_\alpha^4$ ,  $0 < \alpha < 1$ . Denote by  $\mathbf{n}_S$  the normal to the surface  $S$ , directed toward the concavity of  $S$ . Orient the boundary  $L$  so that, when traversing  $L$ , the surface lies on the left. Subject  $S$  to the bushing constraint  $\Sigma$ . Choose, at some point of the boundary  $L$ , that direction of the normal  $\mathbf{n}_\Sigma$  to the surface  $\Sigma$  which makes an acute angle with  $\mathbf{n}_S$ . At the remaining points of  $\Sigma$  continue the direction of the normal by continuity. At every point of the boundary the vectors  $\mathbf{n}_S$  and  $\mathbf{n}_\Sigma$  lie in the normal plane of the curve  $L$ . Denote by  $\varphi$  the angle between  $\mathbf{n}_S$  and  $\mathbf{n}_\Sigma$ . We measure the angle from  $\mathbf{n}_S$  to  $\mathbf{n}_\Sigma$ . We assume here that  $\varphi > 0$  if, looking from the side of the direction of the curve  $L$ , the angle is measured counterclockwise, and  $\varphi < 0$  if the angle is measured clockwise. The angle  $\varphi = \varphi(s)$  is a function of the arc length of the curve  $L$ . Further, denote by  $\psi = \psi(s)$  the angle between the principal normal of the curve  $L$  and the tangential normal  $\eta_S = [\mathbf{n}_S \mathbf{t}]$  of the surface  $S$  along  $L$ . Here  $\mathbf{t}$  is the tangent vector to the curve  $L$ . Obviously,  $0 < \psi(s) < \pi$ .

In what follows we assume that the bushing constraint  $\Sigma$  satisfies the condition

$$-\pi/2 + \psi(s) < \varphi(s) < \pi/2. \quad (2)$$

We shall formulate the main theorems that we have proved.

**Theorem 1.** Let  $S$  be a simply connected surface, and let  $\Sigma$  be an elastic constraint satisfying condition (2). Then:

- a) the surface  $S$  admits exactly three linearly independent infinitesimal (in general, nontrivial) bendings;
- b) the surface  $S$ , fixed at the vertex, admits exactly one linearly independent infinitesimal (in general, nontrivial) bending;
- c) the surface  $S$ , fixed at the vertex together with the tangent plane, admits no infinitesimal bendings, even trivial ones.

Thus, the elastic constraint  $\Sigma$ , in the terminology of I. N. Vekua (<sup>1</sup>), is almost rigid. By imposing on the surface a new point-type constraint, almost rigid constraints can always be made rigid.

**Theorem 2.** Let  $S$  be a simply connected surface, and let  $\Sigma$  be an inelastic bushing satisfying condition (2). Then to each infinitesimal bending of the bushing  $\Sigma$  satisfying on  $L$  the condition  $\vec{\tau}\mathbf{n}_\Sigma \neq 0$ , where  $\vec{\tau}$  is the vector of the displacement field of the points of  $\Sigma$ , there corresponds exactly:

- a) a three-parameter family of infinitesimal bendings of the surface  $S$ ;
- b) a one-parameter family of infinitesimal bendings of the surface  $S$ , fixed at the vertex;
- c) one infinitesimal bending of the surface  $S$ , fixed at the vertex together with the tangent plane.

If on the curve  $L$ ,  $\vec{\tau}\mathbf{n}_\Sigma \equiv 0$ , then the corresponding bushing constraint may be regarded as elastic, and for it Theorem 1 is valid.

Thus, a non-elastic constraint  $\Sigma$ , under the condition  $\vec{\tau}\mathbf{n}_\Sigma \neq 0$ , is always non-rigid, and under the condition  $\vec{\tau}\mathbf{n}_\Sigma \equiv 0$ , is almost rigid.

**Theorem 3.** Let  $S$  be a doubly connected surface, and let  $\Sigma$  be an elastic sleeve satisfying condition (2). Then:

- a) the surface  $S$  can admit no more than one linearly independent infinitesimal (generally speaking, nontrivial) bending;
- b) the surface  $S$ , fixed at some point, admits no infinitesimal bendings, even trivial ones.

**Theorem 4.** Let  $S$  be a doubly connected surface, and let  $\Sigma$  be a non-elastic sleeve satisfying condition (2). Then:

- a) the surface  $S$  admits a one-parameter family of infinitesimal bendings if and only if the sleeve on  $L_1$  or  $L_2$  ( $L = L_1 + L_2$ ) satisfies the condition

$$\oint V(\vec{\tau}\mathbf{n}_\Sigma, s) ds = 0, \quad (3)$$

where  $V$  is a certain known operator;

- b) the surface  $S$ , fixed at some point, admits one nontrivial infinitesimal bending if and only if the sleeve  $\Sigma$  on  $L_1$  or  $L_2$  satisfies condition (3).
4. In <sup>(1)</sup> the general problem of infinitesimal bendings of surfaces under sleeve constraints is formulated. There, infinitesimal bendings of convex surfaces under elastic and non-elastic sleeve constraints are studied in detail under the assumption that the sleeve along  $L$  is orthogonal to the surface  $S$ , i.e.  $\varphi(s) \equiv \pi/2$ . Comparing Theorems 1-4 with the results of I. N. Vekua's investigations, we see that the character of the infinitesimal bendings of the surfaces under consideration, for elastic sleeve constraints satisfying condition (2), is exactly the same as for orthogonal sleeves. Therefore Theorems 1-4 may be formulated for sleeve constraints satisfying condition (2), where equality can be attained only at all points of the boundary simultaneously. The example of I. Kh. Sabitov from work <sup>(2)</sup> shows that the right-hand side of this inequality is sharp, i.e. it cannot be improved while preserving the results of Theorems 1 and 2. As for the left-hand side, for some classes of surfaces it can be improved <sup>(3)</sup>. The character of the infinitesimal bendings of surfaces under non-elastic sleeve constraints differs somewhat from the character of infinitesimal bendings under non-elastic orthogonal sleeves. For example, under non-elastic orthogonal sleeves, assertion c) of Theorem 2 is violated. In this sense the right bound for the variation of the angle  $\varphi(s)$ , obtained in the paper, is sharp. The left-hand side of this inequality can be improved for some surfaces.

A number of paragraphs of the book <sup>(1)</sup> are also devoted to the study of infinitesimal bendings of some convex surfaces under nonorthogonal sleeves of a special kind.

In work <sup>(2)</sup>, I. Kh. Sabitov considered infinitesimal bendings of convex surfaces with a boundary condition of generalized sliding (i.e. the displacement vector  $\vec{\tau}$  of the points of the surface under its infinitesimal bending along the boundary  $L$  satisfies the condition  $\vec{\tau}\mathbf{n}_s = 0$ , where  $\mathbf{n}_s$  is an arbitrarily prescribed continuous vector field). He established theorems analogous to those presented here, under the assumption that the vector field  $\mathbf{n}_s$  differs little (in some norm) from a constant vector. Thus, for the case when the field  $\mathbf{n}_s$  is the field of normals of some surface  $\Sigma$  containing the curve  $L$ , the result of work <sup>(2)</sup> is improved.

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## CITED LITERATURE

<sup>1</sup> I. N. Vekua, *Generalized Analytic Functions*, Moscow, 1959.

<sup>2</sup> I. Kh. Sabitov, DAN, 147, No. 4 (1962).

<sup>3</sup> V. T. Fomenko, DAN, 157, No. 4 (1964).

*Note: Figure translations are in progress. See original paper for figures.*

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