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O. V. SARMANOV

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Abstract

Full Text

O. V. SARMANOV

CHARACTERISTIC COEFFICIENTS OF RANDOM DISTRIBUTIONS

(Presented by Academician S. N. Bernstein on 28 XI 1964)

1. It is known that the power moments

$$m_k = Mx^k = \int_{-\infty}^{\infty} x^k dF(x), \quad k = 1, 2, \dots, \quad (1)$$

unlike the characteristic function $\varphi(t) = Me^{itx}$, are easy to estimate from statistical observations, and in those cases when they exist they characterize the distribution $F(x)$ quite completely. But since for many distributions the moments do not exist, it seems of interest to consider other characteristics which would admit statistical estimation, would exist for all distributions, and, like characteristic functions, would completely determine this distribution.

As such characteristics it is proposed to consider the sequence of complex numbers $\{\lambda(k)\}$ —we shall call them **characteristic coefficients** (c.c.):

$$\lambda(k) = Me^{2ik \operatorname{arctg} x} = \omega(k) + i\tilde{\omega}(k), \quad k = 1, 2, \dots, \quad (2)$$

where

$$\omega(k) = M \cos 2k \operatorname{arctg} x = MT_{2k} \left(\frac{1}{\sqrt{1+x^2}} \right) = M(1+x^2)^{-k} \sum_{l=0}^k (-1)^l C_{2k}^{2l} x^{2l}, \quad (3)$$

$$\tilde{\omega}(k) = M \sin 2k \operatorname{arctg} x = MU_{2k} \left(\frac{1}{\sqrt{1+x^2}} \right) = M(1+x^2)^{-k} \sum_{l=1}^k (-1)^{l-1} C_{2k}^{2l-1} x^{2l-1}, \quad (4)$$

where $T_{2k}(y)$ and $U_{2k}(y)$ are Chebyshev polynomials, respectively of the first and second kind.

With the aid of the known ((1), p. 410) generating functions of the Chebyshev polynomials, the **generating function** of the c.c. is readily found:

$$\psi(u) = M \frac{\frac{1-u^2}{2}(1+x^2) + 2iux}{(1+u)^2x^2 + (1-u)^2} = \frac{1}{2} + \sum_{k=1}^{\infty} \lambda(k)u^k, \quad 0 \leq u < 1. \quad (5)$$

Since $|\lambda(k)| \leq 1$, $\psi(u)$ has no singular points inside the unit circle (we note that by u we have denoted u_1^2 , where $-1 < u_1 < 1$) and

$$\lambda(k) = \frac{\psi^{(k)}(0)}{k!}, \quad k = 1, 2, \dots \quad (6)$$

We shall call the generating function $\psi(u)$ the **Chebyshev transform** of the distribution law $F(x)$ and denote it by $\mathfrak{T}_u(F(x))$. For symmetric distributions $\psi(u)$ is real and has a simpler expression

$$\mathfrak{T}_u(F(x)) = \operatorname{Re} \psi(u) = \frac{1}{2} \frac{1-u}{1+u} M \frac{x^2 + 1}{x^2 + [(1-u)/(1+u)]^2}. \quad (7)$$

Let us give explicit expressions for the Chebyshev transforms, respectively, for the uniform distribution on the interval $[-l, l]$, for the normalized normal distribution, and for the Cauchy distribution:

$$\mathfrak{T}_u\left(\frac{x+l}{2l}\right) = \frac{1}{2} \frac{1-u}{1+u} + \frac{2u}{(1+u)^2} \frac{1}{l} \operatorname{arc} \operatorname{tg} l \frac{1+u}{1-u}, \quad (8)$$

$$\mathfrak{T}_u\left(\frac{1}{2} + \Phi(x)\right) = \frac{1}{2} \frac{1-u}{1+u} + \frac{2u}{(1+u)^2} \sqrt{\frac{\pi}{2}} e^{\frac{1}{2}[(1-u)/(1+u)]^2} \left[1 - 2\Phi\left(\frac{1-u}{1+u}\right)\right], \quad (9)$$

$$\mathfrak{L}_u\left(\frac{\pi}{2} + \operatorname{arc} \operatorname{tg} x\right) = 1/2. \quad (10)$$

As the last formula shows, for the Cauchy distribution all the characteristic coefficients are equal to zero.

2. As is seen from (2), the characteristic coefficients are the values of the characteristic function Me^{itz} of the random variable $z = 2 \operatorname{arc} \operatorname{tg} x$ for $t = 1, 2, \dots$. All values of z are concentrated on the interval $[-\pi, \pi]$; therefore we shall consider in more detail the class of such distributions. For the characteristic functions of these distributions we shall use the special notation

$$\varphi(\pi, t) = \varphi_R(\pi, t) + i\varphi_J(\pi, t) = \int_{-\pi}^{\pi} \cos tx dF(x) + i \int_{-\pi}^{\pi} \sin tx dF(x). \quad (11)$$

As we shall now prove, the characteristic function $\varphi(\pi, t)$ is completely determined by the countable set of values which it assumes at the positive integral points of the real axis, i.e., by the sequence of complex numbers $\{\varphi(\pi, k)\}$, $k = 1, 2, \dots$

In contrast to the general case, we shall also call the sequence $\{\varphi(\pi, k)\}$ the **sequence of characteristic coefficients of a distribution concentrated on the interval** $[-\pi, \pi]$. Note that if a density $p(x)$ of the distribution exists, then the characteristic coefficients differ only by a constant factor from its Fourier coefficients.

We shall proceed from the well-known expansion of $\cos tx$ in a Fourier series for fixed fractional t and for $-\pi \leq x \leq \pi$:

$$\begin{aligned} \cos tx &= \frac{\sin \pi t}{\pi t} + \frac{2t}{\pi} (\sin \pi t) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2 - t^2} \cos kx = \\ &= \frac{\sin \pi t}{\pi t} + \sum_{k=1}^{\infty} \frac{2t}{k+t} \frac{\sin \pi(t-k)}{\pi(t-k)} \cos kx; \end{aligned} \quad (12)$$

the second form shows that formula (2) does not lose its meaning also when t is equal to any positive integer. For fixed t , the series (12) converges uniformly on the whole interval $[-\pi, \pi]$, and it may be integrated term by term on this interval.

In particular, the first of the integrals (11) will take the form

$$\varphi_R(\pi, t) = \frac{\sin \pi t}{\pi t} + \sum_{k=1}^{\infty} \frac{2t}{k+t} \frac{\sin \pi(t-k)}{\pi(t-k)} \varphi_R(\pi, k), \quad (13)$$

where the last formula does not lose its meaning also for t equal to positive integers.

From (12) we immediately obtain

$$\sin tx = \frac{\sin \pi t}{\pi} x + \sum_{k=1}^{\infty} \frac{2t^2}{k(k+t)} \frac{\sin \pi(t-k)}{\pi(t-k)} \sin kx, \quad (12')$$

whence

$$\varphi_J(\pi, t) = \frac{\sin \pi t}{\pi} m_1 + \sum_{k=1}^{\infty} \frac{2t^2}{k(k+t)} \frac{\sin \pi(t-k)}{\pi(t-k)} \varphi_J(\pi, k); \quad (13')$$

let us recall that the first moment m_1 always exists for the class of distributions under consideration.

Formulas (13) and (13') show that the sequence of numbers m_1 and $\{\varphi(\pi, k)\}$, $k = 1, 2, \dots$, completely determines the characteristic function $\varphi(\pi, t)$ on the entire real axis. (Since $\varphi_R(\pi, t)$ is an even function and $\varphi_J(\pi, t)$ is an odd function, it was sufficient to consider the case $t \geq 0$.)

Using the second initial moment m_2 , it is easy to obtain a formula analogous to (13'):

$$\varphi_R(\pi, t) = 1 - \frac{m_2}{2} \frac{t}{\pi} \sin \pi t - \sum_{k=1}^{\infty} \frac{2t^3}{k^2(k+t)} \frac{\sin \pi(t-k)}{\pi(t-k)} [1 - \varphi_R(\pi, k)], \quad (14)$$

in which the terms of the series already decrease as k^{-4} .

These formulas can be used, for example, to check tables of Bessel functions with zero index $J_0(t)$. It is known ⁽²⁾ that

$$J_0(t) = \frac{1}{\pi} \int_{-1}^1 \frac{\cos tx}{\sqrt{1-x^2}} dx, \quad (15)$$

i.e. $J_0(t)$ is the characteristic function of the symmetric distribution law $F(x) = \arcsin x/\pi + 1/2$, $-1 \leq x \leq 1$, and since this interval is contained in the interval $[-\pi, \pi]$, then

$$J_0(t) = \frac{\sin \pi t}{\pi t} + \sum_{k=1}^{\infty} \frac{2t}{k+t} \frac{\sin \pi(t-k)}{\pi(t-k)} J_0(k), \quad (15')$$

$$J_0(t) = 1 - \frac{t}{4\pi} \sin \pi t - \sum_{k=1}^{\infty} \frac{2t^3}{k^2(k+t)} \frac{\sin \pi(t-k)}{\pi(t-k)} [1 - J_0(k)]; \quad (15'')$$

in this case $m_2 = +1/2 = -J_0''(0)$.

Let us also give an expression for the moment m_{2s} through m_2 and the sequence of characteristic coefficients:

$$\frac{m_{2s}}{(2s)!} = \frac{\pi^{2s-2}}{(2s-1)!} \frac{m_2}{2!} + (-1)^s \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} [1 - \varphi_R(\pi, k)]}{k^{2s+1}} \sum_{l=0}^{s-2} (-1)^l \frac{(\pi k)^{2l+1}}{(2l+1)!}. \quad (16)$$

3. Limit theorem. *Let there be a distribution law $F(x)$ with characteristic function $\varphi(\pi, t)$ and an entire sequence of distribution laws $F_n(x)$ with characteristic functions $\varphi_n(\pi, t)$, and let all distributions be concentrated on the interval $[-\pi, \pi]$. If the first moments $m_{1,n}$ and all characteristic coefficients $\varphi_n(\pi, k)$ as $n \rightarrow \infty$ converge respectively to m_1 and to the characteristic coefficients $\varphi(\pi, k)$,*

$k = 1, 2, \dots$, then $\varphi_n(\pi, t) \rightarrow \varphi(\pi, t)$ as $n \rightarrow \infty$ uniformly on any finite interval $|t| < T$.

We give the **proof** for the real part $\varphi_R(\pi, t)$. Let $\varepsilon > 0$ be arbitrarily small and let T , without loss of generality, be taken to be an integer positive number. Consider the remainder of the series (13) for $k > T$ and $|t| < T$:

$$r_T(t) = \sum_{k=T+1}^{\infty} \frac{2}{\pi} \frac{t}{k^2 - t^2} [\sin \pi(k-t)] \varphi_R(\pi, k); \quad (17)$$

since $|\varphi_R(\pi, k) \sin \pi(k-t)| \leq 1$, the series (17) has a majorant common for any $\varphi(\pi, t)$. For sufficiently large k_0 and all $|t| < T$,

$$|r_{k_0}(t)| \leq \sum_{k=k_0}^{\infty} \frac{2}{\pi} \frac{T}{k^2 - T^2} < \frac{\varepsilon}{3}. \quad (18)$$

Consider and estimate the difference $\varphi_R(\pi, t) - \varphi_{n,R}(\pi, t)$ for $|t| < T$:

$$\begin{aligned} |\varphi_R(\pi, t) - \varphi_{n,R}(\pi, t)| &\leq \sum_{k=1}^{\infty} \left| \frac{2t}{k+1} \frac{\sin \pi(k-t)}{\pi(k-t)} [\varphi_R(\pi, k) - \varphi_{n,R}(\pi, k)] \right| \leq \\ &\leq \sum_{k=1}^{k_0-1} \frac{2T}{k} |\varphi_R(\pi, k) - \varphi_{n,R}(\pi, k)| + |r_{k_0}(t)| + |r_{k_0,n}(t)| < \varepsilon, \end{aligned}$$

if n is sufficiently large, since, by virtue of (18), the sum of both remainders is already

less than $2/3 \varepsilon$, and, because of the convergence $\varphi_{n,k}(\pi, k) \rightarrow \varphi_R(\pi, k)$ for $k = 1, 2, \dots, k_0 = 1$, i.e., at a finite number of points, the arbitrary smallness of the first sum follows.

The uniform convergence of the imaginary parts $\varphi_{n,J}(\pi, t)$ to $\varphi_J(\pi, t)$ for $|t| < T$ and $n \rightarrow \infty$ is established analogously; it is only necessary additionally to use the assumed convergence of the first moments.

4. Let us return to the general case of distributions concentrated on the entire real axis. We have already noted that the c.c. are the values at integral points of the characteristic function of the random variable $z = 2 \arctan x$, all values of which are concentrated on the interval $[-\pi, \pi]$. If now the convergence of the c.c. $\{\lambda_n(k)\}$ of some sequence of laws $\{F_n(x)\}$ to the corresponding c.c. $\{\lambda(k)\}$ of the law $F(x)$ takes place, then, by the theorem of the preceding point, the distribution laws of the variables $z_n = 2 \arctan x_n$ converge as $n \rightarrow \infty$ to the distribution law of the variable $z = 2 \arctan x$ (at all points of continuity), whence, by virtue of the monotonicity of the

continuous mapping $z = 2 \arctan x$, the analogous convergence of the distribution law of x_n to the distribution law of x follows. True, one must additionally require also the convergence of the “first moments,” i.e., fulfillment of the condition

$$\mathbf{M} 2 \arctan x_n \rightarrow \mathbf{M} 2 \arctan x$$

as $n \rightarrow \infty$. Thus, the following assertion has been proved:

Main theorem. *The sequence of c.c. defined by formula (2) and $\tilde{m}_1 = \mathbf{M} 2 \arctan x$ completely characterizes any probability distribution; moreover, instead of the sequence (2) one may consider the Chebyshev transform (5).*

Remark 1. The Fourier series for $\sin tx$ with fractional t

$$\sin tx = \frac{2 \sin \pi t}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} k}{k^2 - t^2} \sin kx, \quad -\pi < x < \pi \quad (19)$$

does not converge to the expanded function at $x = \pm\pi$. Therefore, in deriving (13') we proceeded from the expansion (12) for $\cos tx$.

Let us note, however, that from (19) one can obtain the following expression for the first moment:

$$m_1 = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \varphi_J(\pi, k) + (p - q)\pi, \quad (20)$$

where

$$p = \mathbf{P}(x = \pi), \quad q = \mathbf{P}(x = -\pi), \quad (21)$$

i.e., for a complete characterization of the distribution, instead of specifying the first moment m_1 , it is sufficient only to specify the difference $p - q$ and the sequence of c.c. $\{\varphi(\pi, k)\}$, $k = 1, 2, \dots$

Remark 2. The equality to zero of all c.c. of the Cauchy law is easily obtained directly, without the aid of (10). Indeed, the mapping $z = 2 \arctan x$ transforms this distribution into the uniform distribution on the interval $[-\pi, \pi]$ with characteristic function $\sin \pi t / \pi t$, equal to zero for positive integral t ; in formula (13) in this case only the first term remains.

Remark 3. Formulas (13), (14), and (16) may be used as approximations when the main mass of the distribution is concentrated on the interval $[-\pi, \pi]$, which is the case, for example, in the normal distribution with mean 0 and variance 1. In the latter case the indicated formulas give a satisfactory approximation if one restricts oneself to the first four terms of the expansions.

Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.

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