A. V. KUZNETSOV

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Abstract

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A. V. KUZNETSOV

ANALOGUES OF "SHEFFER' S STROKE" IN CONSTRUCTIVE LOGIC

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1.

As is well known, there exists a Boolean function, usually called "Sheffer' s stroke" ([1]), and expressible by the formula $\overline{(p\&q)}$ (or $\overline{pq}$), such that by means of superpositions one can express through this function any Boolean function (i.e., any function of the algebra of logic, or, in other words, any function of the class P_2 ([2])). The question naturally arises whether this situation is preserved in passing from classical logic to constructive (i.e., intuitionistic) logic; whether there are in it analogues of "Sheffer's stroke,"i.e., objects possessing an analogous property. Since in the constructive logic of propositions no functions are usually constructed,* we shall, modifying the formulation of the question accordingly, seek such analogues of "Sheffer' s stroke" that are formulas.

For the sake of completeness, and for use in the proofs, we shall consider, besides constructive and classical logics, also some other propositional logics. However, in order to avoid complications superfluous for the present work, we shall regard a logic as given as soon as the set of all formulas true in it is given (where in some cases truth may be understood as derivability in one or another calculus, and in others as universal validity under one or another interpretation).

2.

Definition of the concept of a **formula** (of propositional logic): the alphabet

$$p\ q\ r\ s\ t\ 0\ 1\ (\ )\ \&\ \vee\ \supset\ \neg;$$

the words p_0, q_0, r_0, s_0, and t_0 are variables (hereafter denoted for short by p, q, r, s, and t, respectively); if A_0 is a variable, then A_{10} is also a variable**; a variable is a formula; if A and B are formulas, then $\neg A$, $(A\&B)$, $(A \vee B)$, and $(A \supset B)$ are formulas.

Considering one or another logic L, we shall denote by $V(L)$ the set of all formulas true in L.

A logic L is called **enumerable** if $V(L)$ is recursively enumerable. A logic L is called **superconstructive** if $V(L)$ is closed under applications of the substitution rules and modus ponens (cf. ([1]), p. 67) and all formulas in the following list belong to $V(L)$:

$$(p \supset (q \supset p)), \quad ((p \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))), \quad ((p\&q) \supset p),$$

$$((p\&q) \supset q), \quad (p \supset (q \supset (p\&q))), \quad (p \supset (p \vee q)), \quad (q \supset (p \vee q)),$$

$$((p \supset r) \supset ((q \supset r) \supset ((p \vee q) \supset r))), \quad ((p \supset q) \supset ((p \supset \neg q) \supset \neg p)),$$

$$(p \supset (\neg p \supset q)).$$

*** Below we consider such logics that are superconstructive and enumerable (s.c.e. logics). Examples of s.c.e. logics are: constructive logic K (the minimal of the s.c.e. logics, i.e., the weakest of them), classical logic K_0 (the minimal among such s.c.e. logics L that $(p \vee p) \in V(L)$), and absolutely contradictory logic $K_{\text{a.p.}}$ (in which all formulas are true).

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* Although analogues of Boolean functions could be introduced, for example, as the results of identifying equivalent formulas considered each with respect to some list of variables (arguments).

** A typical example of a variable: p_{111110}. This notation for variables guarantees the impossibility of one variable being contained in another distinct from it.

*** Compare with the concept of a superconstructive calculus in ([3], [4]).

Many other examples of s.k. p-logics are constructed by means of finite distributive structures (f.d.-structures), i.e., such finite partially ordered sets, each of which is given by a list of its elements and by a table of the partial-order relation ($\leqslant$), and such that for any two of its elements there exist their least upper bound (l.u.b.) and their greatest lower bound (g.l.b.), with the l.u.b. distributive with respect to the g.l.b. Here & is interpreted as the sign of the g.l.b., $\vee$ as the sign of the l.u.b.; the sign 0 denotes the least element of the f.d.-structure, the sign 1 its greatest element; $x \supset y$ is defined as the greatest of such elements z that $(x\&z) \leqslant y$; $\neg x$ as $x \supset 0$. For each f.d.-structure D, the logic $L(D)$ of this structure is defined as the logic in which true are precisely those formulas that, under the above interpretation of the operation signs, are identically equal on D to the element 1. The superconstructivity of the logic $L(D)$ is proved in essence, for example, by Curry ([5]).

3. We call a formula $\mathfrak{A}$ **nonrepeating** if no variable has more than one occurrence in $\mathfrak{A}$. We call a formula $\mathfrak{A}$ ***n*-ary** if the number of distinct variables occurring in $\mathfrak{A}$ is equal to n. The formulas $(p \supset p)$, $(p\&\neg p)$, and $((\mathfrak{A} \supset \mathfrak{B})\&(\mathfrak{B} \supset \mathfrak{A}))$ are denoted briefly by 1, 0, and $(\mathfrak{A} \sim \mathfrak{B})$, respectively.

A finite sequence of formulas $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_n$ is called an **expression** in the logic L through the list (of formulas) Σ if, for each of its terms $\mathfrak{A}_i$, at least one of the following four alternatives holds: 1) $\mathfrak{A}_i$ is a variable; 2) $\mathfrak{A}_i \in \Sigma$; 3) there exist $j, k < i$ and a variable π such that $\mathfrak{A}_i$ is the result of substituting the formula $\mathfrak{A}_j$ into the formula $\mathfrak{A}_k$ in place of all occurrences of π; 4) there exists $l < i$ such that $(\mathfrak{A}_i \sim \mathfrak{A}_l) \in V(L)$. We say that the formula $\mathfrak{A}$ is **expressible** in the logic L through the list Σ if $\mathfrak{A}$ is a term of some expression in L through Σ. We call the list Σ **functionally complete** in the logic L if all formulas are expressible in L through Σ (cf. ([2])). We call a formula $\mathfrak{A}$ **Shefferian** in the logic L if the list consisting of the single $\mathfrak{A}$ is functionally complete in L (an analogue of the "Sheffer stroke" in the logic L).

It is obvious that from expressibility (functional completeness, Shefferianness) in constructive logic there follows expressibility (respectively) in every s.k. p-logic. It is also obvious that the formulas $\neg(p\&q)$ and $\neg(p \vee q)$ are examples of binary nonrepeating formulas Shefferian in K_0.

4. **Theorem 1.** *There exists a five-place nonrepeating formula which is Shefferian in constructive logic. An example of such a formula is* $(p \supset ((q\&\neg r)\&(s \vee t)))$.

Since the list

$$\neg p, \quad (p\&q), \quad (p \vee q), \quad (p \supset q) \tag{1}$$

is functionally complete in the logic K, to prove the theorem it is enough to write an expression in K through the list consisting of the single formula $(p \supset ((q\&\neg r)\&(s \vee t)))$, among whose terms all the terms of the list (1) occur. The expression is: $(p \supset ((q\&\neg r)\&(s \vee t)))$, p, q, s, $(p \supset ((q\&\neg r)\&(s \vee s)))$, $(p \supset ((q\&\neg r)\&s))$, $(p \supset ((q\&\neg r)\&p))$, $(p \supset ((p\&\neg r)\&p))$, $(p \supset \neg r)$, $(p \supset \neg p)$, $\neg p$, $(p \supset \neg\neg p)$, 1, $\neg 1$, 0, $(p \supset ((q\&\neg 0)\&s))$, $(p \supset ((q\&\neg 0)\&1))$, $(p \supset q)$, $(1 \supset ((q\&\neg 0)\&s))$, $(s\&q)$, $(p\&q)$, $(1 \supset ((q\&\neg r)\&(s \vee t)))$, $(1\&((1\&\neg r)\&(s \vee t)))$, $(1 \supset ((1\&\neg 0)\&(s \vee t)))$, $(s \vee t)$, $(p \vee t)$, $(p \vee q)$.

Theorem 2. *There exists a ternary formula which is Shefferian in constructive logic. An example of such a formula is* $(((p \vee q)\&\neg r) \vee (\neg p\&(q \sim r)))$.

The corresponding expression: $(((p \vee q)\&\neg r) \vee (\neg p\&(q \sim r)))$, p, q, r, $((p \vee q)\&\neg q) \vee (\neg p\&(q \sim q)))$, $((p\&\neg q) \vee \neg p)$, $((p\&\neg p) \vee \neg p)$, $\neg p$, $((p\&\neg\neg p) \vee \neg p)$, $(p \vee \neg p)$, $\neg(p \vee \neg p)$, 0, $\neg 0$, 1, $(((p \vee q)\&\neg 0) \vee$

$$\vee(\neg p\&(q \sim 0))),\ ((p \vee q) \vee (\neg p\&\neg q)),\ (((p \vee q)\&\neg 1) \vee (\neg p\&(q \sim 1))),\ (\neg p\&q),\ \neg q,\ (\neg p\&\neg q),\ (\neg p\&r),\ (\neg(\neg p\&\neg q)\&r),\ (\neg(\neg p\&\neg q)\&\&((p \vee q) \vee (\neg p\&\neg q))),\ (p \vee q),\ (((0 \vee q)\&\neg r) \vee (\neg 0\&(q \sim r))),\ ((q\&\neg r) \vee (q \sim r)),\ ((q\&\neg p) \vee (q \sim p)),\ (\neg(\neg p\&q)\&r),\ (\neg(\neg p\&q)\&\&((q\&\neg p) \vee (q \sim p))),\ (p \sim q),\ ((p \vee q) \sim q),\ (p \supset q),\ (p \sim (p \supset q)),\ (p\&q).$$

5. **Theorem 3.** Every formula that is Shefferian in constructive logic has at least five occurrences of variables.

* **Corollary.** There is no repeat-free formula with fewer than five places that is Shefferian in constructive logic.

Theorem 4. If a c.p.c.-logic L is such that $V(L) \neq V(K_{\text{a.p}})$ and $V(L) \neq V(K_0)$, and a formula $\mathfrak{A}$ is two-place and such that $\neg p$ is expressible in L through $\mathfrak{A}$ *, then $(p \supset q)$ is not expressible in L through $\mathfrak{A}$.

Corollary. No two-place formula is Shefferian in constructive logic.

For the proof of Theorems 3 and 4, based on the lemmas below, we shall use two more c.p.c.-logics: **Smetanich' s logic** (briefly, the logic S), defined as the logic of a three-element structure, and the logic $L(D_6)$ of the f.d.-structure D_6, defined as that substructure of the structure of all subsets of the set $\{\alpha, \beta, \gamma, \delta\}$ which has the following list of elements: $\emptyset, \{\alpha\}, \{\alpha, \beta\}, \{\alpha, \gamma\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \delta\}$.
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Lemma 1. Every formula having fewer than five occurrences of variables is expressible in K through one of the following four lists:

$$(p\&q),\ (p \vee q),\ (p \supset q); \qquad (2)$$
$$\neg p,\ (p\&q),\ (p \vee q),\ (\neg p \supset q); \qquad (3)$$
$$\neg p,\ (p\&q),\ (\neg p \vee q),\ (p \supset q); \qquad (4)$$
$$\neg p,\ (\neg p\&q),\ (p \vee q),\ (p \supset q). \qquad (5)$$

Lemma 2. The lists (2), (3), (4), and (5) are not functionally complete in the logics K_0, S, $L(D_6)$, and S, respectively.

Lemma 3. If a c.p.c.-logic L is such that $V(L) \neq V(K_{\text{a.p}})$ and $V(L) \neq V(K_0)$, then $V(L) \subseteq V(S)$.

Lemma 4. If a formula $\mathfrak{A}$ is two-place, and $\neg p$ is expressible in the logic S through $\mathfrak{A}$, then $(p \supset q)$ is not expressible in S through $\mathfrak{A}$.

6. Lemma 1 is proved by estimating the number of occurrences of the signs $\&, \vee$, and $\supset$, and by considering cases distinguished according to whether the sign $\neg$ occurs in the formula and, if it does, in what place. Lemma 3 is also easily proved, if one observes that the result of substituting the formulas $0, 1$, and $(p \vee \neg p)$ into any formula in the places of all variables is equivalent in K to one of these same three formulas. For the proof of Lemmas 2 and 4, however, there are used a generalization of the notion of formula and the apparatus of preservation formulas, described below.

The definition of a **generalized formula** differs from the definition of an (ordinary) formula given at the beginning of §2 in that two new signs are also included in the alphabet: $\uparrow$ and $\downarrow$, and a new formation rule for formulas is added: if A is a formula, then $\uparrow A$ and $\downarrow A$ are formulas. Below, the term "formula" will mean a generalized formula wherever generalized formulas are meaningful and it is not stipulated that the formula is ordinary. Moreover, the signs $\uparrow$ and $\downarrow$ are interpreted only in the logics of f.d.-structures.

Let the list of elements of an f.d.-structure have the form: $a_0, a_1, \dots, a_{k-1}$. Then we put: $\uparrow a_i = a_{i+1}$, where addition is modulo k; $\downarrow a_i = 1$, if $a_i = 1$; $\downarrow a_i = 0$, if $a_i \neq 1$ $(i = 0, 1, \dots, k-1)$. It is not hard to prove that every operation that can be defined on the set $\{a_0, a_1, \dots, a_{k-1}\}$ can be expressed by some (generalized) formula.

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* That is, through a list consisting of $\mathfrak{A}$ alone.

** Cf. ([6]), pp. 363-364, and also ([4]) (the structures I_1 and J_2).

Considering a formula $\mathfrak{A}$ with respect to a list of variables $\pi_1, \pi_2, \dots, \pi_n$, we shall denote by $\mathfrak{A}(\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n)$ the result of substituting the formulas $\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n$ into the formula $\mathfrak{A}$ in the places of the variables $\pi_1, \pi_2, \dots, \pi_n$, respectively.

A **formula of preservation** of a formula $\mathfrak{A}$ by a formula $\mathfrak{B}$ is any formula which, if the formula $\mathfrak{A}$ is considered with respect to some list of variables $\pi_1, \dots, \pi_n$ containing all the variables occurring in $\mathfrak{A}$, and the formula $\mathfrak{B}$ with respect to some list $\rho_1, \dots, \rho_m$ with the analogous condition, and if all the variables σ_i^j $(i = 1, \dots, m;\ j = 1, \dots, n)$ are pairwise distinct, has the following form:

$$\mathfrak{A}[\sigma_1^1, \dots, \sigma_n^1] \supset \mathfrak{A}[\sigma_1^2, \dots, \sigma_n^2] \supset (\dots \supset (\mathfrak{A}[\sigma_1^m, \dots, \sigma_n^m] \supset\subset \mathfrak{A}[\mathfrak{B}[\sigma_1^1, \sigma_1^2, \dots, \sigma_1^m], \dots, \mathfrak{B}[\sigma_n^1, \sigma_n^2, \dots, \sigma_n^m]])\infty))).$$

We say that the formula $\mathfrak{B}$ **preserves** the formula $\mathfrak{A}$ in the logic L if the formula of preservation of the formula $\mathfrak{A}$ by the formula $\mathfrak{B}$ is valid in L.* We say that the formula $\mathfrak{B}$ is **separated** in the logic L by the formula $\mathfrak{A}$ from the list Σ, if all its members preserve the formula $\mathfrak{A}$ in L, whereas $\mathfrak{B}$ does not preserve it.

Lemma A. *If a formula $\mathfrak{B}$ is separated in the logic L from the list Σ by some formula, then $\mathfrak{B}$ is not expressible in L through Σ.*

For the proof of Lemma 2 it remains to note that the formula $(p \supset q)$ is separated in the logic S from the list (3) by the formula $((p \supset q)\&(\neg p \supset \neg q))$, the formula $(p \vee q)$ is separated in $L(D_6)$ from (4) by the formula $\neg p$, and the formula $(p\&q)$ is separated in S from (5) by the formula $(((p \vee \neg p) \vee q)\&(\neg p \sim \neg q))$. After this, Lemma 4 also follows from the fact that every two-place ordinary formula through which the formula $\neg p$ is expressible in S preserves in S the formulas $(\neg\neg p \supset p)$ and $(\neg p \sim \neg q)$ and does not preserve the formulas $\neg p$ and $\neg\neg p$, and therefore, as is seen from consideration of the tables possible for it in S, is expressible through (3).

7. For evaluating the method of formulas of preservation the following is useful.

Theorem 5. *For every f.d.-structure D, every generalized formula $\mathfrak{A}$, and every list Σ of generalized formulas, one and only one of the following two alternatives holds: either $\mathfrak{A}$ is expressible in $L(D)$ through Σ, or $\mathfrak{A}$ is separated in $L(D)$ from Σ by some generalized formula.*

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CITED LITERATURE

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* Cf. the notion of preservation of a predicate in (7), pp. 104–105. The subtle difference between preservation of a formula and preservation of a predicate is connected with the existence of f.d.-structures D and ordinary formulas $\mathfrak{A}$ and $\mathfrak{B}$ such that the formula $(\mathfrak{A} \supset \mathfrak{B})$ is not valid in $L(D)$, but for all D-values of the variables belonging to it, from the fact that $\mathfrak{A} = 1$ it follows that $\mathfrak{B} = 1$. An example of such $D, \mathfrak{A}$, and $\mathfrak{B}$ was communicated to the author by V. A. Yankov; in this case it turned out that the formula $\neg p$ does not preserve in $L(D)$ the formula $((\neg\neg p \supset p) \vee \neg p)$, but does preserve the predicate $((\neg\neg p \supset p) \supset (p \vee \neg p)) = 1$. As V. P. Dulub informed the author, in the logics of the structures of Jaśkowski (see (4)) analogous examples do not exist.

Note: Figure translations are in progress. See original paper for figures.

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