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Abstract

Full Text

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THEORY OF LINEAR POLYNOMIAL OPERATIONS ON TOPOLOGICAL GROUPS

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1°. We shall consider functions defined on a bicompact commutative group G. Since the functions $\{e^{ikx}\}_{k=-\infty}^{\infty}$ are the characters of the circle group, the trigonometric case of the theory of linear polynomial operations, which has been studied in many works, is a special case of our considerations when the group G is the circle group.

A natural generalization of the notion of an ordinary trigonometric polynomial is given by finite linear combinations of characters of the group G. We shall assume that an invariant integration is defined in G and that the measure $\mu(G) = 1$. Then the characters $\{\chi_k\}$ of the group G form an orthonormal system of functions with respect to the measure μ*

$$\int \chi_k(t)\overline{\chi_j(t)}\, d\mu(t) = \delta_{kj}.$$

A generalized trigonometric polynomial on the group G is a function of the form

$$P(x) = \sum_{k=1}^{m} \lambda_k \chi_k(x), \qquad x \in G, \tag{1}$$

where $\{\lambda_k\}_{k=1}^{m}$ are complex numbers. The collection of characters $\{\chi_j(x)\}_{j=1}^{m}$ will be denoted by $C(P)$ and called the **spectrum** of the polynomial P. Let P be a fixed generalized trigonometric polynomial and Q an arbitrary generalized trigonometric polynomial. We say that Q has spectrum P if $C(Q) \subset C(P)$. The set of all generalized trigonometric polynomials with spectrum $C(P)$ will be denoted by $\Pi(P)$.

By $L^1 = L^1(G)$ we denote the totality of all μ-measurable functions for which $\int |f|\, d\mu(t) < \infty$. In what follows we use additive notation for the operation in G.

By $f_t(x)$ we denote the function $f(x+t)$, where $t \in G$.

Consider a functional space $E = E(G)$, which is defined by the following axioms:

1. The elements of E are functions from L^1.
2. E is a linear normed space. Here addition of functions and multiplication of a function by a number are defined in the usual way.
3. If $f \in E$, then $f_t \in E$ for every $t \in G$, and moreover $\|f_t\| \le \|f\|$.
4. E contains the set of all generalized trigonometric polynomials.

Definition. Let E_1 and E_2 be spaces of type E. We shall say that $U = U(f)$ is a generalized linear trigonometric poly-

* In what follows it is assumed that the integral is taken over the whole group.

polynomial operation with spectrum $C(P)$ on the group G, if: 1) $U(f)$ is a linear operation from E_1 into E_2; 2) for every $f \in E_1$, $U(f) \in \Pi(P)$, where P is a given fixed generalized trigonometric polynomial on the group G of the form (1). The set of all such U will be denoted by $\mathfrak{M}(P)$. By $\mathfrak{N}(P)$ we shall denote the subset of $\mathfrak{M}(P)$ consisting of operators that commute with translation. Thus, if $U \in \mathfrak{N}(P)$, then for any $f \in E_1$ and any $t \in G$ the equality

$$U(f_t) = (U(f))_t$$

holds.

2°. Let $U \in \mathfrak{M}(P)$. Introduce the operator

$$\widetilde{U}(f, x) = \int U(f_t, x - t)\, d\mu(t). \tag{2}$$

Theorem 1. *The following assertions hold:*

1. *For every $U \in \mathfrak{M}(P)$, $\widetilde{U} \in \mathfrak{N}(P)$.*
2. *If $U \in \mathfrak{N}(P)$, then $\widetilde{U} = U$.*
3. $\|\widetilde{U}\| \le \|U\|$.

Proof. Properties 1-2 are obvious. Let us prove property 3. The right-hand side of equality (2) may be regarded as an abstract integral of Bochner type. It is known that for such integrals the inequality[(1)]

$$\left\| \int f\, d\mu \right\| \le \int \|f\|\, d\mu$$

holds.

Therefore, from axiom 3 and equality (2) it follows that

$$\|\widetilde{U}(f)\| \le \|U\|\, \|f\|.$$

Consequently, $\|\widetilde{U}\| \leq \|U\|$.

Theorem 2. *Let $U \in \mathfrak{M}(P)$; then for any $f \in E_1$ the identity*

$$\widetilde{U}(f - S(f, P)) = 0$$

holds, where $S(f, P)$ is the partial sum of the Fourier series of the function f with spectrum $C(P)$:

$$S(f, P) = \sum_{\chi \in C(P)} C_\chi \chi, \qquad C_\chi = C_\chi(f) = \int f \overline{\chi} \, d\mu(t).$$

Proof. Put $\varphi = f - S(f, P)$ and

$$\psi(x) = \widetilde{U}(\varphi, x).$$

Compute the integral

$$I_\chi = \int \psi(x) \overline{\chi}(x) \, d\mu(x),$$

where $\chi \in C(P)$. By the invariance of integration and Fubini' s theorem, we have

$$I_\chi = \int \overline{\chi}(z) \, d\mu(z) \int U(\varphi_t, z) \overline{\chi}(t) \, d\mu(t). \tag{3}$$

Since the operator and the integral are interchangeable[1,2], it follows that

$$\int U(\varphi_t, z) \overline{\chi}(t) \, d\mu(t) = U\left(\int \varphi_t \, \overline{\chi}(t) \, d\mu(t), z \right). \tag{4}$$

We also note that, according to the definition of φ and $S(f, P)$,

$$\int \varphi_t \, \overline{\chi}(t) \, d\mu(t) = 0.$$

Therefore equality (4) implies that

$$\int U(\varphi_t, z) \overline{\chi}(t) \, d\mu(t) = 0.$$

for any $\chi \in C(P)$. Consequently, by virtue of equality (3), $I_\chi = 0$ for any $\chi \in C(P)$. Since $C(\psi) \in C(P)$, it follows that $\psi(x) = 0$ for any $x \in G$.

Theorem 3. Let $U_i \in \mathfrak{M}(P)$, $i = 1, 2$, and suppose that on the set $\Pi(P)$ the equality

$$U_1(Q) = U_2(Q), \qquad Q \in \Pi(P). \tag{5}$$

holds. Then, for any $f \in E_1$, the equality

$$\widetilde{U}_1(f) = \widetilde{U}_2(f). \tag{6}$$

holds.

Proof. It follows from Theorem 2 that, for any $f \in E_1$,

$$\widetilde{U}_i(f) = \widetilde{U}_i[S(f, P)], \qquad i = 1, 2. \tag{7}$$

Since $S(f, P) \in \Pi(P)$, equality (5) implies that

$$U_1[S(f, P)] = U_2[S(f, P)], \tag{8}$$

and therefore

$$\widetilde{U}_1[S(f, P)] = \widetilde{U}_2[S(f, P)].$$

From (7) and (8), (6) follows.

Corollary. Let the operation $U \in \mathfrak{M}(P)$ be such that there exists a convolution of the form

$$\sigma(f, x) = \int f(x + t) K(t)\, d\mu(t),$$

where $K \in \Pi(P)$, coinciding with the operation U on $\Pi(P)$. Then, for any $f \in E_1$, the equality

$$\widetilde{U}(f) = \sigma(f). \tag{9}$$

holds.

Proof. Since, by assumption,

$$U(Q) = \sigma(Q), \qquad Q \in \Pi(P),$$

then, according to Theorem 3,

$$\widetilde{U}(f) = \tilde{\sigma}(f), \qquad f \in E_1. \tag{10}$$

It is easy to see that $\tilde{\sigma} = \sigma$. Therefore (9) follows from (10).

Remark. In defining the space E, it was not assumed that the axiom on the density of polynomials in E holds. Therefore, in particular, in this note (unlike in the works ([3,4])) equality (9) is established without the mentioned axiom.

Theorem 4. Let $U \in \mathfrak{M}(P)$. Then, for any $f \in E_1$ and any $x \in G$, the equality

$$\widetilde{U}(f, x) = \int f(x+t)\widetilde{U}\left(\sum_{\chi \in C(P)} \chi, -t\right) d\mu(t).$$

holds.

This theorem is easily obtained from Theorem 2.

In conclusion, we note that the special cases of Theorems 1-4, when $E_1 = E_2 = \widetilde{C}$—the space of continuous 2π-periodic functions and G is the group of rotations of the circle—were considered in the works ([5,6]).

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Note: Figure translations are in progress. See original paper for figures.