MATHEMATICS

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Abstract

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ON GENERAL ERGODIC THEORY

1. The questions considered in this article originate in the works of J. von Neumann. We have in mind his theorem on a one-parameter group of unitary operators and their averages over long intervals ([1]). Further investigations were carried out by F. Riesz, Isida, Kakutani, and others.

M. Day ([2]) showed that every bounded commutative semigroup of operators in a reflexive Banach space has averages converging to a projector. Finally, Eberlein ([3]) dispensed with the requirement of reflexivity of the space and proved that every bounded commutative semigroup of operators in an arbitrary Banach space is ergodic. One of the results given in the present note is that every bounded commutative semigroup of operators in an arbitrary (complete) locally convex space is ergodic. The basis of this fact is a generalization of Cesàro' s theorem to a semigroup of operators with an arbitrary cardinality of the system of generators. These and other results are obtained on the basis of the theory of semifields ([4,5]). In using a number of notions and definitions we follow Eberlein ([3]).

Let X be a space weakly normed over the semifield R_Δ ([4]), and let G be a semigroup of linear operators acting in X:

$$G = \{T;\ T:\ X \to X\}.$$

By the convex hull over the set G we shall mean

$$G^* = \left\{T^*;\ T^* = \sum_{i=1}^{n} a_i T_i,\ a_i \geqslant 0,\ \sum_{1}^{n} a_i = 1,\ T_i \in G\right\},$$

where n is an arbitrary positive integer.

Introduce the notation

$$O(x) = \{T^* x;\ T^* \in G^*\}, \qquad x \in X.$$

Definition 1. A family of linear mappings $\{A_\mu\}$ of the space X into itself **forms a basic Ω-system*** with respect to the semigroup G, if the following conditions are satisfied:

I. For any $x \in X,\ T \in G$,

$$\|TA_\mu x - A_\mu x\|_{\mu\in\Omega} \to 0; \qquad \|A_\mu Tx - A_\mu x\|_{\mu\in\Omega} \to 0.$$

Here, of course, the norm is meant to take values in K_Δ, where K_Δ is the cone in R_Δ.

Next we define the norm of an operator. By the **norm of an operator** A we shall mean the mapping $\|A\| = \nu : K_\Delta \to K_\Delta$, defined for any $a \in K_\Delta$ by the equality

$$\nu(a) = \sup_{\|x\|\leqslant a} (\|Ax\|) \tag{1}$$

* Here we change Eberlein' s terminology. By Ω we mean a set whose elements are all possible finite subsets of Δ.

and satisfying the conditions

$$\begin{gathered} v(\lambda x) = \lambda v(x) \quad \text{for } \lambda \geqslant 0, \quad x \in K_\Delta, \\ v(x+y) \leqslant v(x) + v(y), \quad x \in K_\Delta,\ y \in K_\Delta. \end{gathered} \tag{2}$$

Without dwelling on many other (quite analogous to the classical ones) properties of the operator norm proposed by I. M. Dekhtyarev (communicated to me in an oral conversation while discussing questions connected with the present article), we formulate the second condition imposed on the family of operators $\{A_\mu\}$.

II. The norm of the family of operators $\{A_\mu\}$ is bounded, i.e. there exists a mapping $v : K_\Delta \to K_\Delta$, satisfying conditions (2) and such that $v_\mu = \|A_\mu\| \leqslant v$ for every $\mu \in \Omega$. The latter means that for every $a \in K_\Delta$

$$v_\mu(a) = \sup_{\|x\|\leqslant a} \{\|A_\mu x\|\} \leqslant v(a) \quad \text{for every } \mu \in \Omega.$$

III. For every $x \in X$ and all $\mu \in \Omega$:

$$A_\mu x \in \overline{O}(x),$$

where $\overline{O}(x)$ is the closure of the set $O(x)$.

Definition 2. A semigroup G of linear operators will be called **ergodic** if it admits at least one basic Ω-system.

2. Let $\mathfrak{A} = \{T_q\},\ q \in \Delta$, be a collection of linear commutative operators acting in the space X, and let G be the semigroup generated by the operators from $\mathfrak{A}$. For every $T \in G$ suppose that the relation

$$\|T\| \leqslant v. \tag{3}$$

is fulfilled.

Let now λ be an element of Ω, more precisely, $\lambda = (q_1, q_2, \dots, q_s)$.

Theorem 1. *The family $\{A_{\lambda,t}\}$, where*

$$A_{\lambda,t} = \frac{1}{R(t)} \sum_{m_1+\ldots+m_s=0}^{t} T_{q_1}^{m_1} T_{q_2}^{m_2} \ldots T_{q_s}^{m_s},$$

i.e. the Cesàro-type mean, forms a basic Ω-system with respect to the semigroup G; here the numbers s and t must satisfy the relation $s/t \to 0$ as $t \to \infty$, and $R(t)$ is the number of terms entering into the sum written on the right in the expression for $A_{\lambda,t}$.

Theorem 2. *The semigroup G, generated by the operators from $\mathfrak{A}$, is ergodic.*

Corollary. If G is a commutative semigroup with a system of generators $\mathfrak{A}$ of arbitrary cardinality, then, by specifying a one-to-one mapping f of the set $\mathfrak{A}$ onto the set of indecomposable idempotents Δ of the semilattice R_Δ (the cardinality of Δ is assumed equal to the cardinality of $\mathfrak{A}$) and by indexing each element of $\mathfrak{A}$ by that $q \in \Delta$ which corresponds to it under the mapping f, according to Theorem 2 we shall have:

Theorem 3. *If the semigroup G, generated by a system of generators $\mathfrak{A}$ of arbitrary cardinality, is commutative and bounded, then it is ergodic.*

In this case the basic Ω-system is the family $\{A_{\lambda,t}\}$ constructed above, with $T_q \in \mathfrak{A},\ q \in \Delta$.

Theorem 4. If the semigroup G is ergodic, x is some element of the space X, and $\{A_\mu\}$ is a basic Ω-system with respect to G, then the conditions (a), (b), (c) given below stand in the relation $(a) \sim (b) \to (c)$:

(a)
$$\lim_{\mu\in\Omega} A_\mu x = x_0.$$

(b) $x_0 \in \overline{O}(x)$ and $Tx_0 = x_0$ for all $T \in G$.

(c) For any $y, z \in X$ such that $y, z \in \overline{O}(x)$, one has

$$\|A_\mu y - A_\mu z\|_{\mu\in\Omega} \to 0.$$

The assertion $(a) \sim (b)$, contained in Theorem 3.1 of Eberlein, is easily proved also with the aid of norms with values in K_Δ. As for the relation $(c) \to (a)$, its validity requires additional information about the family of operators $\{A_\mu\}$.

Definition 3. An element $x \in X$ will be called **ergodic with respect to the semigroup** G **with limit point** x_0, if the equality

$$x_0 = \lim_{\mu \in \Omega} A_\mu x,$$

holds, where $\{A_\mu\}$ is a basic Ω-system with respect to the semigroup G.

We shall denote by T_Ω the mapping of X into X defined by the equality

$$\lim_{\mu \in \Omega} A_\mu x = T_\Omega x \qquad \text{for all } x \in X.$$

Theorem 5. Let X be a complete space normed over R_Δ. If G is ergodic, then the set of all ergodic elements forms a closed linear subspace E of the space X, invariant with respect to the semigroup G. Moreover, the operator T_Ω is linear on E, $T_\Omega^2 = T_\Omega$, and $\|T_\Omega\| \le \nu$, if $\|A_\mu\| \le \nu$ for all $\mu \in \Omega$.

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Note: Figure translations are in progress. See original paper for figures.