ON THE DIMENSION OF GROUPS WITH A LEFT-INVARIANT TOPOLOGY

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Abstract

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The article studies the construction of topological groups proposed by A. A. Markov (^{2}) and Kakutani (^{1}), in order to clarify the relation between the dimensions of homogeneous spaces—groups with a non-invariant topology—under different definitions of dimension.

Let X be an arbitrary set. By $G(X)$ we denote the free algebraic group generated by the set X, and by $A(X)$ the free algebraic abelian group generated by the set X.

Lemma 1. *Let ρ be a pseudometric on the set X. There exist invariant pseudometrics $\bar{\rho}$ on the groups $G(X)$ and $A(X)$, satisfying the following conditions: if $x, y \in X$, and e is the neutral element of the group, then $\bar{\rho}(x, e) = 1$, $\bar{\rho}(x, y) = \rho(x, y)$. Among all such invariant pseudometrics there exists a largest one (i.e., one taking the largest value). Among all left-invariant pseudometrics satisfying the same conditions there exists a largest one. The indicated largest pseudometrics are constructed effectively. If ρ is a metric, then the indicated pseudometrics are also metrics.*

We shall carry out the proof for left-invariant pseudometrics. The remaining cases are considered in the paper of M. I. Graev (^{3}).

Every element $z \in G(X)$ has an irreducible representation $z = (x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \dots, x_n^{\varepsilon_n})$, $x_i \in X$, $\varepsilon_i = \pm 1$. From two words $Z_1 = (x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n})$ and $z_2 = (y_1^{\eta_1}, \dots, y_m^{\eta_m})$ one can form a third $Z_3 = (x_1^{\varepsilon_1}, \dots, x_n^{\varepsilon_n}, y_1^{\eta_1}, \dots, y_m^{\eta_m})$. Then we write $Z_3 = Z_1 Z_2$. If $Z = Z_1 Z_2 \dots Z_n$, then we shall say that a **decomposition** τ of the word Z is constructed. Let a decomposition τ of the word $Z = Z_1 Z_2 \dots Z_n$ be given, where each Z_i consists of one letter or has the form $Z_i = x^{-1} y^1$. Such a decomposition will be called **proper**, and the words Z_i will be called **elementary words**. If Z_i is an elementary word, then set $\|Z_i\| = \Delta$ if Z_i consists of one letter, and $\|Z_i\| = \rho(x, y)$ if $Z_i = x^{-1} y^1$. Let τ be a proper decomposition of the word Z,

$$Z = Z_1 \dots Z_n.$$

Set

$$\|Z\|_\tau = \sum_{i=1}^{n} \|Z_i\|.$$

Define the pseudometric on $G(X)$ by the equalities: $\bar{\rho}(z, e) = \min\{\|Z\|_\tau\}$, where Z is the irreducible representation of the element z, and τ runs through all proper decompositions of the word Z;

$$\bar{\rho}(z_1, z_2) = \bar{\rho}(z_2^{-1} z_1, e).$$

The pseudometric $\bar{\rho}$ constructed is the largest left-invariant metric compatible with the pseudometric ρ. Further, we shall denote this pseudometric by ρ_ℓ, and the largest pseudometric among invariant pseudometrics will be denoted by ρ_i.

Let $Y_i = X \cup \{e_i\} \cup X^{-1}$ be a copy of the discrete union of three sets: X, a copy X^{-1}, and the point $\{e_i\}$. If a pseudometric is given on X, then on Y_i we also define a pseudometric by setting the distances between points from different sets equal to one, and on X and X^{-1} equal to the distance in the original pseudometric. By Z denote the direct sum of the spaces Y_i with fixed point $\{e_i\}$. In Z define a pseudometric by the equality

$$\rho(z', z'') = \max_i \rho(z_i', z_i'')\, i^2.$$

There exist natural mappings of the space Z onto the groups $G(X)$ and $A(X)$.

Theorem 1. Let ρ be a pseudometric in X. The mappings $f : Z \to G(X)$, $g : Z \to A(X)$ are uniformly continuous if in $G(X)$ and $A(X)$ one defines the pseudometrics ρ . These mappings are uniformly open if in $G(X)$ and $A(X)$ one defines the pseudometrics $(\sqrt{\rho})$. If on X a system of pseudometrics R is given, then on $G(X)$ we construct the corresponding systems of pseudometrics R and R .

Lemma 2. Let $G_n \subset G(X)$ be the set of all elements having a representation of length $\leq n$. If the system of pseudometrics R generates a separated topology, then G_n are closed with respect to the topologies generated by the systems R .

Lemma 3. Under the same assumptions G_n is topologically equivalent to a discrete union of sets of the form

$$X \times X \times \cdots \times X,$$

where there are n factors, in which, possibly, some diagonals are contracted to points.

Lemma 4. If in $G(X)$ one introduces the metrics R , then instead of spaces of the form $X \times X \times \cdots \times X$ in Lemma 3 one should take the sets $X \times X \cdots \times X$ with a finer topology.

This topology is defined as follows: X is either in its usual topology or in the discrete one; suppose that on $X \times \cdots \times X$, with $n - 1$ factors, the topology has already been defined; then in $X \times \cdots \times X$, with n factors, layers of the form

$X \times \cdots \times X \times \{x\}$ are glued along the diagonal, and on the diagonal the topology of X is taken (usual or discrete).

Lemma 5. If X is paracompact, then the spaces $X \times \cdots \times X$ described in Lemma 4 are paracompact.

Theorem 2. If X is paracompact in the system of pseudometrics R, then $G(X)$ is paracompact in the system of pseudometrics R .

Theorem 3. If the topological products $X \times \cdots \times X$ are paracompact, then $G(X)$ is paracompact in the system of pseudometrics R .

Theorem 4. Let a system of pseudometrics R be given in X, and a system R be given in $G(X)$. If X is normal, then the equality

$$\dim X = \dim G(X)$$

holds.

Dimension is to be understood as the dimension of the Čech extension of the space.

Corollary. For every metric space X there exists a homogeneous metric space of the same dimension dim, containing the space X.

Theorem 5. Let a system of pseudometrics R be given in X, and a system R be given in $G(X)$. The inequality

$$\operatorname{ind} G(X) \leq \operatorname{ind} X + 1$$

is valid.

Theorem 6. Let X be metric and $\operatorname{ind} X = 0$. X is embedded in a metric group G, zero-dimensional in the sense of ind, if and only if in $X \times X$ there is a base of neighborhoods with empty boundary.

The following questions remain open. Does there exist a metric group whose dimensions dim and ind do not coincide? Does there exist a metric space X for which $\dim X - \operatorname{ind} X > 1$? Can a complete metric space be embedded in a complete metric group with preservation of dimension?

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CITED LITERATURE

1. S. Kakutani, *Proc. Imp. Acad. Tokyo*, **20**, 595 (1944).

2. A. A. Markov, *Izv. AN SSSR, ser. matem.*, **9**, No. 1 (1945).

3. M. I. Graev, *Izv. AN SSSR, ser. matem.*, **12**, No. 3 (1948).

Note: Figure translations are in progress. See original paper for figures.

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