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Abstract

Full Text

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DIFFRACTION OF A PLANE WAVE BY A SEGMENT

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1. We consider the stationary problem of diffraction of the plane wave $u_{\text{inc}} = e^{ik(x\sin\beta + y\cos\beta)}$ by the segment AB ($A:\ x=-\delta,\ y=0,\quad B:\ x=\delta,\ y=0$), on which the boundary condition $\partial u/\partial n\big|_{AB} = 0$ is imposed. The solution can be written in the form $u(k,x,y) = u_{\text{g}}(k,x,y) + u_{\text{dif}}(k,x,y)$, where u_{g} is the solution of the geometrical-optics approximation, i.e., the sum of the incident and reflected waves, and u_{dif} satisfies the Sommerfeld radiation condition. We shall study the asymptotics, in the far zone, of the function $u_{\text{dif}}(k,x,y)$: $\lim_{\rho\to\infty}\sqrt{\rho}\,e^{-ik\rho}u_{\text{dif}}(k,x,y) = U(\beta,\theta,k)$, where $x=\rho\sin\theta,\ y=-\rho\cos\theta$ (see Fig. 1). For arbitrary values of β and θ we shall give the asymptotics of $U(\beta,\theta,k)$ as $k\delta\to\infty$ to within any prescribed power $(k\delta)^{-N}$.

It is easy to show that $u_{\text{dif}}(k,x,y)$ is an odd function of y. Therefore $U(\beta,\theta,k)$ is extended in an odd way through the values $\theta=\pm\pi/2$: $U(\beta,\pm\pi/2+\varphi,k) = -U(\beta,\pm\pi/2-\varphi,k)$. Similarly, $U(\pm\pi/2+\psi,\theta,k) = -U(\pm\pi/2-\psi,\theta,k)$. As a result, we may restrict ourselves to considering the values $|\beta|<\pi/2$ and $|\theta|<\pi/2$, bearing in mind that through the values $\beta=\pm\pi/2$ and $\theta=\pm\pi/2$ the function $U(\beta,\theta,k)$ must be continued in an odd way.

Fig. 1

We shall first give formulas for the asymptotic expansion of $U(\beta,\theta,k)$ in powers of $(k\delta)^{-1}$. These formulas are not applicable when $|\beta|\to\pi/2$ or $|\theta|\to\pi/2$ (i.e., when the direction of incidence β or the direction of observation θ approaches the direction grazing along AB), since the coefficients of the asymptotic expansion of $U(\beta,\theta,k)$ tend to infinity as $|\beta|\to\pi/2$ or $|\theta|\to\pi/2$.

Therefore, for each pair of limiting values β,θ (where $\beta=\pm\pi/2,\ \theta=\pm\pi/2$) we shall give formulas that also provide an arbitrarily accurate asymptotic approximation to $U(\beta,\theta,k)$, but are applicable in a neighborhood of the limiting values under consideration. These approximations will be expressed in terms of the Fresnel integral

$$F(\varphi) = \frac{2}{\sqrt{\pi i}}\int_0^{2\sqrt{k\delta}\,\sin(\pi/4-\varphi/2)} e^{is^2}\,ds \tag{1}$$

and functions which, in the neighborhood under consideration of the values β, θ, have asymptotic expansions in powers of $(k\delta)^{-1}$ with bounded coefficients. Each term of the resulting series will be continued through the limiting values β, θ under consideration in an odd way.

2. We shall give the expansion of $U(\beta, \theta, k)$ in powers of $(k\delta)^{-1}$. Consider all possible broken lines (optical paths) in which the first link is a ray arriving from the direction β and ending at one of the endpoints of the segment AB, all remaining links (except the last) coincide with AB, and the last link is a ray beginning at one of the endpoints of AB and going in the direction θ. To each optical path S there corresponds a contribution U_S to $U(\beta, \theta, k)$, so that

$$U(\beta, \theta, k) = U_A + U_B + U_{AB} + U_{BA} + U_{ABA} + U_{BAB} + U_{ABAB} + \cdots, \tag{2}$$

where the index of each term on the right-hand side indicates the sequence

vertices in the optical path corresponding to this term. In formula (2)

$$U_A(\beta, \theta, k) = U_B(-\theta, -\beta, k) = \sqrt{\frac{2i}{\pi k\delta}}\, e^{ik\delta(\sin\theta - \sin\beta)} \frac{\sin(\pi/4 - \beta/2)\sin(\pi/4 + \theta/2)}{\sin\beta - \sin\theta}, \tag{3}$$

and each of the remaining terms U_S has the asymptotic expansion:

$$U_S \simeq 2\sqrt{\delta}\, e^{ik\delta[2(n-1) - \sin\psi - \sin\varphi]} \times \tag{4}$$

$$\times \sum_{s=0}^{\infty} \left(\frac{i}{4k\delta}\right)^{s+n/2} \sum_{\substack{p,q=0 \\ p+q=s}} \frac{\alpha^{(n)}_{s,p,q}}{[\cos(\pi/4 + \psi/2)]^{2p+1}[\cos(\pi/4 + \varphi/2)]^{2q+1}}.$$

Here n is the number of vertices in the optical path S corresponding to the function U_S; $\psi = \beta$ if S begins at the vertex A, and $\psi = -\beta$ if S begins at the vertex B; $\varphi = -\theta$ if S ends at the vertex A, and $\varphi = \theta$ if S ends at the vertex B. The coefficients $\alpha^{(n)}_{s,p,q}$ are symmetric in p and q and are determined by the recurrence formula:

$$\alpha^{(n+1)}_{s,p,q} = \sum_{\chi=q}^{s-p} \sum_{\sigma=0}^{\chi-q} \frac{(-1)^{s-\chi}\Gamma(s-\chi+1/2)\Gamma(s-\chi-p+\sigma+1/2)}{\pi\Gamma(\sigma+1/2)\Gamma(s-\chi-p+1)}\, \alpha^{(n)}_{\chi,\sigma,q},$$

$$\alpha^{(2)}_{s,p,q} = \frac{(-1)^{s+1}\Gamma(s+1/2)\Gamma(s-p-q-1/2)}{2\sqrt{2}\,\pi^2\Gamma(s-p-q+1)}. \tag{5}$$

Fig. 2

Figure 1: Fig. 2

3. Formulas (3)—(5) make it possible to determine the asymptotic expansion of $U(\beta, \theta, k)$ in powers of $(k\delta)^{-1}$ up to any fixed power $(k\delta)^{-N}$. To do this, in the right-hand side of (2) one must take $4N$ terms corresponding to optical paths containing no more than $2N$ vertices, and for each such term compute, by formulas (4), (5), the required number of terms of its asymptotic expansion. The resulting expression $U_N(\beta, \theta, k)$ will satisfy the reciprocity principle.

However, the formulas obtained in this way are not applicable for β and θ close to $\pm\pi/2$. Indeed, as $\varphi \to \pi/2$ the coefficients of expansion (4) tend to infinity, while through the value $\varphi = -\pi/2$ they continue in an even manner. Recall that here $\varphi = \pm\theta$. An analogous situation occurs when $|\beta| \to \pi/2$. In order to obtain formulas for $U(\beta, \theta, k)$ applicable for $|\beta|, \ |\theta| \to \pi/2$, it is necessary, for each pair β, θ of limiting values $\beta = \pm\pi/2, \ \theta = \pm\pi/2$, to group the terms in the right-hand side of (2) and jointly consider the contribution of each such group to $U(\beta, \theta, k)$. Since $U(\beta, \theta, k) = U(-\beta, -\theta, k)$, it is sufficient to consider the values $\beta > 0$, i.e., two pairs of limiting values of β, θ: $\beta = \pi/2, \ \theta = \pi/2$ and $\beta = \pi/2, \ \theta = -\pi/2$.

Fig. 2

4. Let us first consider a neighborhood of the values $\beta = \pi/2, \ \theta = \pi/2$. Represent $U(\beta, \theta, k)$ in the form

$$\begin{aligned} U(\beta, \theta, k) = {} & [U_A + U_B + U_{AB}] + [U_{BA} + U_{BAB} + U_{ABA} + U_{ABAB}] \\ & + [U_{BABA} + U_{BABAB} + U_{ABABA} + U_{ABABAB}] + \dots, \end{aligned} \tag{6}$$

where the law by which the following groups are formed is evident. Such a grouping has a simple geometric meaning. In one group are collected those terms whose optical paths, for θ, β close to $\pi/2$, have approximately equal lengths. In the first group are collected the terms whose optical paths pass through the segment AB once; in the second group, the terms passing through AB 3 times, and so on (see Fig. 2).

In any closed neighborhood of the values $\beta = -\pi/2, \ \theta = -\pi/2$, containing no points at which $\beta = -\pi/2$ or $\theta = -\pi/2$, the following formulas hold:

$$U_A + U_B + U_{AB} = F(\theta) U_A + F(\beta) U_B + U''_{AB}, \tag{7a}$$

$$\begin{aligned} U_{B_{lA}} + U_{AB_{lA}} + U_{B_{lAB}} + U_{AB_{lAB}} = {} & F(\theta) F(\beta) U_{B_{lA}} + F(\beta) U'_{B_{lAB}} \\ & + F(\theta) U''_{AB_{lA}} + U'''_{AB_{lAB}}. \end{aligned} \tag{7b}$$

Here $l = 1, 2, \ldots$ is the number of vertices A in the optical path corresponding to the function $U_{B_l A}$: $U_{B_1 A} = U_{BA}$, $U_{B_2 A} = U_{BABA}$, etc. In formulas (7), $F(\varphi)$ is the Fresnel integral (1).

The asymptotic expansion of U''_{AB} has the form

$$U''_{AB} \approx -\frac{\sqrt{\delta}}{\sqrt{2\pi^2}}\, e^{ik\delta(2-\sin\beta-\sin\theta)} \times$$

$$\times \sum_{s=0}^{\infty} \left(\frac{-i}{4k\delta}\right)^{s+1} \sum_{p,q=0} \frac{\Gamma(s+p+q+3/2)}{\Gamma(s+p+q+3)} \sin^{2p+1}\left(\frac{\pi}{4}-\frac{\beta}{2}\right) \sin^{2q+1}\left(\frac{\pi}{4}-\frac{\theta}{2}\right). \tag{8}$$

The asymptotic expansions of the functions $U'_{B_{lAB}}$, $U''_{AB_{lA}}$, $U'''_{AB_{lAB}}$ are determined by the expansion (4) of the function $U_{B_{lA}}$ (in which one must put $n = 2l$, $\varphi = -\theta$ and $\psi = -\beta$) or, more precisely, by the coefficients specifying this expansion,

$$\alpha^{(2l)}_{s,p,q} = \gamma_{s,p,q}.$$

$$U'_{B_{lAB}}(\beta,\theta,k) \approx 2\sqrt{\delta}\, e^{ik\delta(4l-\sin\theta+\sin\beta)} \sum_{s=0}^{\infty} \left(\frac{i}{4k\delta}\right)^{s+l+1/2} c'_s\left(\frac{\pi}{4}-\frac{\theta}{2}, \frac{\pi}{4}-\frac{\beta}{2}\right);$$

$$U''_{AB_{lA}}(\beta,\theta,k) = U'_{B_{lAB}}(\theta,\beta,k); \tag{9}$$

$$U'''_{AB_{lAB}}(\beta,\theta,k) \approx 2\sqrt{\delta}\, e^{ik\delta(4l+2-\sin\beta-\sin\theta)} \sum_{s=0}^{\infty} \left(\frac{i}{4k\delta}\right)^{s+l+1} c''_s\left(\frac{\pi}{4}-\frac{\theta}{2}, \frac{\pi}{4}-\frac{\beta}{2}\right),$$

where

$$c'_s(\varphi,\psi) = \sum_{\chi=0}^{s} \sum_{\sigma=0}^{\chi} \frac{(-1)^{s-\chi} A(s-\chi,\sigma,\varphi)}{\pi} \sum_{p=0}^{\chi-\sigma} \frac{\gamma_{\chi,\sigma,p}}{(\cos\psi)^{2p+1}}, \tag{10a}$$

$$c''_s(\varphi,\psi) = \sum_{\chi=0}^{s} \frac{(-1)^{s-\chi}}{\pi^2} \sum_{m=\chi}^{s} \sum_{\sigma,p=0}^{\sigma+p=\chi} A(s-m,p,\varphi)\, A(m-\chi,\sigma,\psi)\, \gamma_{\chi,\sigma,p}, \tag{10b}$$

$$
\begin{aligned}
A(s,\sigma,\varphi) &= \frac{\Gamma(s+1/2)}{\Gamma(\sigma+1/2)} \sum_{q=0}^{\infty} \frac{\Gamma(s+\sigma+q+3/2)\sin^{2q+1}\varphi}{\Gamma(s+q+2)} \\
&= \sin\varphi \, \frac{\Gamma(s+1/2)\Gamma(s+\sigma+3/2)}{\Gamma(\sigma+1/2)\Gamma(s+1)} \int_0^1 \frac{(1-t)^s \, dt}{(1-t\sin^2\varphi)^{s+\sigma+3/2}}.
\end{aligned} \tag{11}
$$

These formulas make it possible to determine $U(\beta,\theta,k)$ in a neighborhood of the values $\beta=\pi/2$ and $\theta=\pi/2$ with accuracy up to any order $(k\delta)^{-N}$. To do this one must take N groups in the right-hand side of (6) and, for each such group, determine by formula (7b) its contribution to $U(\beta,\theta,k)$. It is easy to see that the expression obtained is continued in an odd manner through the values $\beta=\pi/2$ and $\theta=\pi/2$ and satisfies the reciprocity principle.

5. In a neighborhood of the values $\beta=\pi/2,\ \theta=-\pi/2$ the following grouping of terms in the right-hand side of (2) is carried out:

$$
\begin{aligned}
U(\beta,\theta,k) = U_A &+ [U_B + U_{AB} + U_{BA} + U_{ABA}] \\
&+ [U_{BAB} + U_{ABAB} + U_{BABA} + U_{ABABA}] + \cdots .
\end{aligned} \tag{12}
$$

The optical path corresponding to U_A, for β,θ close respectively to $\pi/2$ and $-\pi/2$, does not pass through AB even once; the paths corresponding to the functions from the first group pass through AB twice, etc. In this formula

$$
U_B + U_{BA} + U_{AB} + U_{ABA}
$$

$$
= F(\beta)F(-\theta)U_B + F(\beta)U'_{BA} + F(-\theta)U''_{AB} + U'''_{ABA}, \tag{13a}
$$

where

$$
\begin{aligned}
U'_{BA}(\beta,\theta,k) = U''_{AB}(-\theta,-\beta,k) &\simeq -\frac{\sqrt{2\delta}\cos(\pi/4-\beta/2)e^{ik\delta(2+\sin\theta+\sin\beta)}}{\pi^{3/2}} \\
&\times \sum_{s=0}^{\infty} \left(\frac{-i}{4k\delta}\right)^{s+1} \sum_{\sigma=0}^{\infty} A\left(s,\sigma,\frac{\pi}{4}+\frac{\theta}{2}\right) \sin^{2\sigma}\left(\frac{\pi}{4}-\frac{\beta}{2}\right);
\end{aligned} \tag{14a}
$$

$$
\begin{aligned}
U'''_{ABA} &\simeq \frac{\sqrt{2\delta}}{\pi^{5/2}} e^{ik\delta(4+\sin\theta-\sin\beta)} \\
&\times \sum_{s=0}^{\infty} \left(\frac{i}{4k\delta}\right)^{s+3/2} (-1)^{s+1} \sum_{m=0}^{s} \Gamma\left(s-m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right) \\
&\times \sum_{q,r=0}^{\infty} F(s-m+q+1, m+r+1) \sin^{2q+1}\left(\frac{\pi}{4}-\frac{\beta}{2}\right) \sin^{2r+1}\left(\frac{\pi}{4}+\frac{\theta}{2}\right).
\end{aligned}
$$

Here $F(q,r)$ are the coefficients of the expansion of

$$\frac{\sqrt{1-x}\sqrt{1-y}}{1-x-y}$$

in powers of x, y:

$$\sum_{q,r=0}^{\infty} x^q y^r F(q,r) = \frac{\sqrt{1-x}\sqrt{1-y}}{1-x-y}. \tag{15}$$

Any other quadruple $U_{B_{lB}} + U_{B_{lBA}} + U_{AB_{lB}} + U_{AB_{lBA}}$ (where $l = 1, 2, \ldots$ is the number of vertices A in the optical path corresponding to the function $U_{B_{lB}}$: $U_{B_1B} = U_{BAB}$, $U_{B_2B} = U_{BABAB}$, etc.) gives the following contribution to the asymptotics of $U(\beta, \theta, k)$:

$$F(\beta)F(-\theta)U_{B_{lB}} + F(\beta)U'_{B_{lBA}} + F(-\theta)U''_{AB_{lB}} + U'''_{AB_{lBA}}, \tag{13b}$$

where $U_{B_{lB}}$ has the asymptotic expansion (3) (in which $n = 2l+1$, $\psi = -\beta$, $\varphi = \theta$), specified by the coefficients $\gamma_{s,p,q} = \alpha^{(2l+1)}_{s,p,q}$, while the functions $U'_{B_{lBA}}$, $U''_{AB_{lB}}$, $U'''_{AB_{lBA}}$ have the asymptotic expansions

$$U'_{B_{lBA}} \simeq 2\sqrt{\delta}\, e^{ik\delta[4l+2+\sin\theta+\sin\beta]} \sum_{s=0}^{\infty} \left(\frac{i}{4k\delta}\right)^{s+l+1} c'_s\left(\frac{\pi}{4}+\frac{\theta}{2}, \frac{\pi}{4}-\frac{\beta}{2}\right); \tag{16a}$$

$$U''_{AB_{lB}} \simeq 2\sqrt{\delta}\, e^{ik\delta[4l+2-\sin\theta-\sin\beta]} \sum_{s=0}^{\infty} \left(\frac{i}{4k\delta}\right)^{s+l+1} c'_s\left(\frac{\pi}{4}-\frac{\beta}{2}, \frac{\pi}{4}+\frac{\theta}{2}\right); \tag{16b}$$

$$U'''_{AB_{lBA}} \simeq 2\sqrt{\delta}\, e^{ik\delta[4l+4+\sin\theta-\sin\beta]} \sum_{s=0}^{\infty} \left(\frac{i}{4k\delta}\right)^{s+l+3/2} c''_s\left(\frac{\pi}{4}+\frac{\theta}{2}, \frac{\pi}{4}-\frac{\beta}{2}\right), \tag{16c}$$

where $c'_s(\varphi, \psi)$ and $c''_s(\varphi, \psi)$ are determined by formulas (10).

6. The results presented were obtained in considering the nonstationary problem of diffraction of a plane wave by AB. In this case each term U_S on the right-hand side of (2) is the far-zone asymptotics of the Fourier transform with respect to t of a cylindrical wave u_S, arising under the successive diffraction of the incident wave and of the cylindrical waves excited by it on the sequence of vertices of the segment AB that form the optical path S.

From each such wave u_S it proved possible to isolate a function which, after the Fourier transform, gives a Fresnel integral in such a way that the remainder has a uniformly convergent ray expansion as θ, β tend to the selected limiting values. The Fourier transform of these remainders then gave us the functions U', U'', U''', and, after grouping the waves according to (6) or (12), it proved possible to write the contribution of each group in the closed form (7) or (13) (depending on which pair of limiting values we consider). Analogous results can be obtained for the boundary condition on the segment AB of the form $u|_{AB} = 0$.

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Note: Figure translations are in progress. See original paper for figures.

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