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Abstract

Full Text

MATHEMATICS

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METRIC PROPERTIES OF PLANE UNIVALENT MAPPINGS OF CLOSED REGIONS

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1°. Classes of mappings. A real function $f(x, y)$ belongs to the class BL_k , $\in BfL_k$ (with respect to the rectangular coordinate system x, y), in a domain D of the plane $z = x + iy$, if f is continuous in D ,

$$\iint_D \left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right] dx dy \leq k < \infty$$

and the derivatives are understood in the sense of S. L. Sobolev. $f \in BL$, if $f \in BL_k$ in D for some $k < +\infty$.

Let $w = T(z) \equiv f_1(x, y) + if_2(x, y)$ be a mapping of D into the plane $w = u + iv$. $T \in BL$ in D , if $f_j \in BL$ in D ; $T \in BL_k$, if, in addition,

$$\iint_D \sum_{j=1}^2 \text{grad}^2 f_j dx dy \leq k < \infty.$$

In what follows we consider only topological mappings.

2°. Suppose that $D \ni 0$, $T(D) = \Delta$, the domains D and Δ are bounded, and $T(0) = 0$. Let \tilde{D} be the closure of D by prime ends K . Carathéodory with the natural topology (see, for example, ⁽¹⁾) and $\rho(t_1, t_2; \tilde{D})$ the relative distance (according to ⁽²⁾) between t_1 and t_2 , internal points or prime ends of D (defined on the basis of the Euclidean metric). The distance $\rho(t_1, t_2; \tilde{\Delta})$ in $\tilde{\Delta}$ has an analogous meaning. The topology determined by the metric ρ in \tilde{D} is equivalent to the topology of \tilde{D} . It is known ^(1,2) that if T and $T^{-1} \in BL$ in D and Δ , respectively, then T and T^{-1} can be extended to $\partial\tilde{D} = \tilde{D} \setminus D$ and $\partial\tilde{\Delta} = \tilde{\Delta} \setminus \Delta$, so that T realizes a homeomorphism $T(\tilde{D}) = \tilde{\Delta}$. From the results of ^(1,2) it follows

Theorem 1 (on the distortion of relative distances). *Let the domains D_T and Δ_T contain the fixed circles $|z| < \delta_0$ and $|w| < \delta_1$, respectively, be bounded, and let $T(D_T) = \Delta_T$, where T and $T^{-1} \in BL_k$ in D_T and Δ_T . If*

$$\varphi_1(\sigma) \equiv \sqrt{3} \exp(-4\pi k \sigma^{-2}), \quad \varphi_2(\sigma) \equiv (4\pi k)^{1/2} \ln^{-1/2} \sqrt{3} \sigma^{-1}$$

and

$$a = \sqrt{3} \min \left\{ \frac{1}{4} \delta_0^2, \frac{1}{4} \delta_1^2, \exp[-4^3 \pi k (3\delta_1)^{-1}] \right\},$$

then, for

$$0 < \rho(t_1, t_2; \tilde{D}_T) \leq a \tag{1}$$

we have

$$\varphi_1[\rho(t_1, t_2; \tilde{D}_T)] < \rho[T(t_1), T(t_2); \tilde{\Delta}_T] < \varphi_2[\rho(t_1, t_2; \tilde{D}_T)]. \tag{2}$$

3°. Consider the set of density classes in \tilde{D}_T and $\tilde{\Delta}_T$ and define, as usual, the distance between density classes $\alpha(M_1, M_2; \tilde{D}_T)$ and $\alpha(N_1, N_2; \tilde{\Delta}_T)$, setting them equal to the deviations of any two sets $M_1, M_2 \subset \tilde{D}_T$ and $N_1, N_2 \subset \tilde{\Delta}_T$ belonging to the corresponding classes

* Instead of the notations $D, \Delta, T(D) = \Delta$, here and below we use the notations $D_T, \Delta_T, T(D_T) = \Delta_T$.

density ((³), pp. 166–167). Then Theorem 1 can be formulated in the following equivalent form*:

Theorem 2 (on the distortion of deviations of sets). Under the conditions and in the notation of Theorem 1 we have: if the closed sets $F_1, F_2 \subset \tilde{D}_T$ satisfy the condition

$$0 < \alpha(F_1, F_2; \tilde{D}_T) \leq \alpha,$$

then

$$\varphi_1[\alpha(F_1, F_2; \tilde{D}_T)] < \alpha[T(F_1), T(F_2); \tilde{\Delta}_T] < \varphi_2[\alpha(F_1, F_2; \tilde{D}_T)].$$

4°. Theorems 1, 2 and their concretizations and refinements can serve as a convenient means for establishing numerous metric-geometric facts about classes of mappings. Theorems 3–6—consequences of Theorem 1—are obtained without invoking any additional properties of the mappings.

Denote by LC the length of the curve C , by SA the area of the set A , and by $A(p, E)$ the subset of points p of the set A satisfying condition E .

Theorem 3 (on uniform continuity and openness of mappings inside domains of definition). Let $\alpha < a$ and $\alpha_1 < \min\{\alpha, \varphi_2^{-1}[\varphi_1(\alpha)]\}$. Then for every point $z_0 \in \widetilde{D}_T [z : \rho(z, \partial\widetilde{D}_T; \widetilde{D}_T) \geq \alpha]$ and all points z' :

$$|z' - z_0| \leq \alpha_1, \quad \varphi_1(|z' - z_0|) < |T(z') - T(z_0)| < \varphi_2(|z' - z_0|),$$

and, consequently, the image of the disk with center at z_0 and radius α_1 covers the disk with center at $T(z_0)$ and radius $\varphi_1(\alpha_1)$ and is covered by the disk of radius $\varphi_2(\alpha_1)$.

Remark 1. Theorem 3 establishes facts of the type of internal covering theorems for conformal mappings, or better, gives moduli of uniform continuity and openness with respect to the origin of coordinates inside the domains of definition of the mappings in the sense of Definitions 1 and 2 of paper ⁽²⁾** . The significance of Theorem 3 is that it gives information sufficient for detecting a connection between the concept of uniform convergence of a sequence of mappings inside the kernel of a sequence of domains and the concept of convergence of the sequence of image domains to a kernel in the sense of C. Carathéodory (see ⁽⁴⁾).

5°. If $G \ni 0$ is a bounded simply connected domain, then put

$$G(\alpha_1, \alpha_2) \equiv G[z : \alpha_1 \leq \rho(z, \partial G; G) \leq \alpha_2], \quad C_\alpha \equiv G(\alpha, \alpha) \quad \text{and} \quad G(\alpha) \equiv G(0, \alpha).$$

Under the conditions of Theorem 1 we have:

Theorem 4 (on the distortion of sets of distance levels). If

$$\alpha_1 \leq \alpha_2 < \alpha, \quad \text{then} \quad T[D_T(\alpha_1, \alpha_2)] \subset \Delta_T[\varphi_1(\alpha_1), \varphi_2(\alpha_2)].$$

Theorem 5 (on the distortion of areas of boundary rings). If, in addition to the conditions of Theorem 1, one requires that all domains Δ_T belong to the disk $|w| < M$, then for $\alpha < a$

$$\pi\delta_1^2 - \pi[\delta_1 - \varphi_1(\alpha)]^2 < ST[D_T(\alpha)] < \pi M^2 - \pi[\delta_1 - \varphi_2(\alpha)]^2,$$

and for a given α one can find a T from the class under consideration such that

$$ST[D_T(\alpha)] > \pi(M^2 - \delta_1^2).$$

If, instead of uniform boundedness of the domains Δ_T , one requires that the boundaries Δ_T be rectifiable and that their lengths not exceed the number L_0 , then

$$ST[D_T(\alpha)] < L_0\varphi_2(\alpha) - \pi\varphi_2^2(\alpha).$$

* In this formulation somewhat more information of a geometric character is contained explicitly.

** Taking the opportunity, we note that these definitions in ⁽²⁾ should be made more precise by adding to them the words “with respect to the origin of coordinates” and by requiring that the continua \tilde{D}_T participating in the definitions contain the point O . The properties of mappings described by Definitions 1 and 2 are subsequently established in ⁽²⁾ precisely in this sense.

Remark 2. The set $C_\alpha = D(\alpha, \alpha)$ is the set of points of the domain D for which

$$\rho(z, \partial\bar{D}; \bar{D}) = \alpha$$

(the level set α of the distance ρ). Let \bar{C}_α be the set of points of D whose Euclidean distance to the boundary D is equal to α . Between C_α and \bar{C}_α there is a connection, which is as follows.

Let $\alpha < \delta_0$. Remove from \bar{C}_α all components that can be separated from 0 by a cut of the domain D with Euclidean diameter $< \alpha$. We obtain the set \bar{C}'_α . Consider all those components of the set \bar{C}'_α which are separated from 0 by a cut of D of diameter α and determine two domains in the plane. We enlarge each such component by adding to it all points of that domain of the plane which belongs to D . The remaining components of \bar{C}'_α are not changed. If \bar{C}_α is the set \bar{C}'_α enlarged in this way, then $C_\alpha = \bar{C}_\alpha$. Taking this into account, Theorems 4 and 5 can also be formulated for the level set of the ordinary distance *.

Remark 3. In particular, those components of the sets C_α and \bar{C}_α which separate 0 and $\partial\bar{D}$ always coincide ($\alpha < a$). Denote such a component by C_α^0 . The curve $T(C_\alpha^0) \subset \Delta_T^0$, generally speaking, is not rectifiable. However, by Theorem 4,

$$T(C_\alpha^0) \subset \Delta_T^0,$$

where Δ_T^0 is a component of the set $\Delta_T[\varphi_1(\alpha), \varphi_2(\alpha)]$, and a certain metric characteristic for $T(C_\alpha^0)$ is the quantity

$$\widehat{LT}(C_\alpha^0),$$

equal to the half-sum of the lengths of the inner and outer boundaries of Δ_T^0 (lengths in the sense of ⁽⁵⁾). From ⁽⁵⁾ it follows that

$$\widehat{LT}(C_\alpha^0) < \frac{\pi d_T^2}{3} \left[\frac{1}{\varphi_1(\alpha)} + \frac{1}{\varphi_2(\alpha)} \right],$$

where d_T is the Euclidean diameter of Δ_T .

6°. Theorem 6 (on the distortion of boundary arcs). Under the hypotheses of Theorem 1 and for $\delta < a$, fix a prime end $t_0 \in \partial \bar{D}_T$ and consider all $t \in \partial \bar{D}_T$ for which

$$\rho(t, t_0; \bar{D}_T) = \delta.$$

The set of such t forms two arcs (segments of prime ends, possibly degenerating into points): t'_-, t''_-, t'_+, t''_+ . Let the ends of these arcs be denoted so that the prime ends $t''_-, t'_-, t_0, t'_+, t''_+$ form a cyclic sequence **. Then:

1. The arc

$$\sigma = T(t''_-)T(t''_+)$$

can be separated from 0 by a cut of the domain Δ_T of Euclidean diameter

$$< 2\varphi_2(\delta),$$

while the arc

$$\tau = T(t'_-)T(t'_+)$$

cannot be separated from 0 by a cut of diameter

$$< \varphi_1(\delta).$$

2. If Δ_T is a disk $|w| < 1$, then τ covers an arc with midpoint at $T(t_0)$ and length

$$> 2r \arcsin \frac{\varphi_1(\delta)}{2},$$

while the arc σ is covered by an arc with midpoint at $T(t_0)$ and length

$$< 2 \arcsin \varphi_2(\delta).$$

3. If for Δ_T there exists a monotone function $\beta(\delta)$, $\beta(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, such that the length of the boundary arc formed by the points t satisfying

$$\rho(t, t_0; \lambda_T) < \delta$$

does not exceed $\beta(\delta)$, then τ covers an arc of length

$$< 2\varphi_1(\delta)$$

and σ is covered by an arc of length

$$< \beta[2\varphi_2(\delta)]$$

***.

Remark 4. All estimates in Theorems 1–6 are of an equi-continuous character (φ_1 and φ_2 do not depend on the admissible domains and admissible mappings).

* The fact that in doing so some components of C_α are excluded corresponds to the nature of things, for the images of such components will be arbitrarily

close to the boundary Δ_T if the components themselves are located in a subdomain D_T not containing 0 and connected with the remaining part of D_T by a sufficiently narrow "neck." Therefore such components cannot participate in estimates of equi-continuous character.

** As is known, $\partial\bar{D}_T$ is cyclically ordered.

*** Example: if Δ_T is convex and is contained in the disk $|w| < M$, then equicontinuously (with respect to T) one will have

$$\beta(\delta) < 48M/(2\delta_1 - \delta)$$

(a rough estimate).

Remark 5. From ^(1, 2) there easily follows a general theorem on the distortion of relative distances in closed domains under mappings of the class \widetilde{BL}_k (see ⁽²⁾). In this case the domains D_τ and $\Delta_\tau \ni \infty$ may also be unbounded. The resulting two-sided estimate of distortion, in the general case, is not of an equally sharp character, but subclasses of mappings are readily singled out for which equal sharpness of the estimates does occur. For all such subclasses there are results of the type of the results of Theorems 1-6. In addition, in ⁽¹⁾ and especially in ⁽²⁾ many concretizations and refinements of the basic estimates are given (refinements for φ_1 and φ_2), so that for conformal and Q -quasiconformal mappings, as well as in the case of domains with additional restrictions on the boundary, substantial refinements are possible. Here, for the sake of brevity in the formulations, we have restricted ourselves to mappings of the class BL_k and do not dwell on these refinements. Let us note in conclusion that for spatial mappings as well, for which estimates of the type (1), (2) have been established (see ⁽⁶⁾), applications of these estimates of the same kind are possible.

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