Academician of the Academy of Sciences of the Moldavian SSR V. A. ANDRUNAKIEVICH, Yu. M. RYABUKHIN

1964

SovietRxiv

Abstract

Full Text

Academician of the Academy of Sciences of the Moldavian SSR V. A. ANDRUNAKIEVICH, Yu. M. RYABUKHIN

MODULES AND RADICALS

In the present note it is shown that the general theory of radicals of associative rings, developed in the works of Kurosh ([1]), Amitsur ([2]), and others, can be set out in an external way—in the language of modules or, what is the same, in the language of representations. In such an exposition an essential role is played by general modules, which generalize the notions of irreducible and primary modules ([3,4]).

Let A be an arbitrary associative ring. By an A-module we shall mean a right A-module. The **annihilator** of an A-module $\mathfrak{M}$ is the set $(0 : \mathfrak{M})_A = \{x \mid x \in A,\ \mathfrak{M}x = 0\}$. If $(0 : \mathfrak{M})_A = 0$, then $\mathfrak{M}$ is a faithful A-module; if $\mathfrak{M}A = 0$, then $\mathfrak{M}$ is a trivial A-module. Let Σ_A be an arbitrary class of A-modules. The **kernel of the class** Σ_A is the intersection

$$\operatorname{Ker} \Sigma_A = \bigcap \{(0 : \mathfrak{M})_A \mid \mathfrak{M} \in \Sigma_A\}.$$

If $\Sigma_A = \emptyset$, then, as usual, we put $\operatorname{Ker} \Sigma_A = A$. If $\operatorname{Ker} \Sigma_A = 0$, then the class Σ_A is called **faithful**. Recall the following well-known fact.

Proposition 1. Let A be an arbitrary ring, B an ideal of the ring A. If $\mathfrak{M}$ is an A/B-module, then, under the composition $xa = x(a + B)$, $\mathfrak{M}$ becomes an A-module, and $B \subseteq (0 : \mathfrak{M})_A$. Conversely, if $\mathfrak{M}$ is an A-module and the ideal $B \subseteq (0 : \mathfrak{M})_A$, then, under the composition $x(a + B) = xa$, $\mathfrak{M}$ becomes an A/B-module. Every submodule of the A/B-module $\mathfrak{M}$ is a submodule of the A-module $\mathfrak{M}$, and conversely. Finally,

$$(0 : \mathfrak{M})_{A/B} = (0 : \mathfrak{M})_A / B.$$

To each ring A we assign a certain class Σ_A of nontrivial A-modules (possibly empty), and let Σ be the class of all Σ_A, where A ranges over the class of associative rings.

Taking Proposition 1 into account, we give the following

Definition 1. A class Σ will be called a **general class of modules** if the following conditions are satisfied:

P.1. From $\mathfrak{M} \in \Sigma_{A/B}$ it follows that $\mathfrak{M} \in \Sigma_A$.

P.2. From $\mathfrak{M} \in \Sigma_A$ it follows that $\mathfrak{M} \in \Sigma_{A/B}$, if $B \subseteq (0 : \mathfrak{M})_A$.

P.3. If the class Σ_A is faithful, then $\Sigma_B \neq \emptyset$ for every nonzero ideal B of the ring A.

P.4. If $\Sigma_B = \emptyset$ for every nonzero ideal B of the ring A, then the class Σ_A is faithful.

It is easy to verify, for example, that the class of all irreducible modules is general. Recall that a nontrivial A-module $\mathfrak{M}$ is called irreducible if the only nonzero submodule in $\mathfrak{M}$ is $\mathfrak{M}$ itself.

Definition 2. Let Σ be a general class of modules. The **Σ-radical** $R(\Sigma, A)$ of the ring A is the kernel $\operatorname{Ker}\Sigma_A$ of the class Σ_A, i.e.

$$R(\Sigma, A) = \operatorname{Ker}\Sigma_A = \bigcap\{(0 : \mathfrak{M})_A \mid \mathfrak{M} \in \Sigma_A\}.$$

If $\Sigma_A = \emptyset$, then the ring A is called **Σ-radical**. If the class Σ_A is faithful, i.e. $R(\Sigma, A) = 0$, then the ring A is called **Σ-semisimple**.

We note that the ring A will be Σ-radical if and only if $A = R(\Sigma, A)$. Indeed, if the ring A is not Σ-radical, i.e. $\Sigma_A \neq \emptyset$, then for any $\mathfrak{M} \in \Sigma_A$ we obtain $R(\Sigma, A) \subseteq (0 : \mathfrak{M})_A \subset A$, since we consider only nontrivial A-modules. Conversely, if the ring A is Σ-radical, i.e. $\Sigma_A = \emptyset$, then $R(\Sigma, A) = \operatorname{Ker}\Sigma_A = A$.

Proposition 2. A homomorphic image of a Σ-radical ring is a Σ-radical ring.

This assertion follows directly from the definition of a Σ-radical ring and condition P.1.

Proposition 3. If B is an ideal of the ring A such that $B \subseteq R(\Sigma, A)$, then

$$R(\Sigma, A/B) = R(\Sigma, A)/B.$$

Indeed, if $\Sigma_A = \emptyset$, then, by Proposition 2, we obtain $R(\Sigma, A(B)) = A/B = R(\Sigma, A)/B$. If, however, $\Sigma_A \neq \emptyset$, then from $B \subseteq R(\Sigma, A) \subseteq (0 : \mathfrak{M})_A$ and conditions P.1, P.2 and Proposition 1 we obtain

$$R(\Sigma, A/B) = \bigcap\{(0 : \mathfrak{M})_{A/B} \mid \mathfrak{M} \in \Sigma_{A/B}\} = \bigcap\{(0 : \mathfrak{M})_A/B \mid \mathfrak{M} \in \Sigma_A\}$$
$$= (\bigcap\{(0 : \mathfrak{M})_A \mid \mathfrak{M} \in \Sigma_A\})/B = R(\Sigma, A)/B.$$

Corollary 1. $R(\Sigma, A/R(\Sigma, A)) = 0$, i.e. the factor ring of any ring by its Σ-radical is Σ-semisimple.

Corollary 2. A ring A is Σ-radical if and only if it is not mapped homomorphically onto nonzero Σ-semisimple rings.

Proposition 4. In any ring A, its Σ-radical $R = R(\Sigma, A)$ coincides with the intersection of all ideals T_α such that the factor rings by them are Σ-semisimple.

Proof. Since $R(\Sigma, A/R) = 0$, it follows that $R \supseteq \bigcap T_\alpha$. Now let $A_\alpha = A/T_\alpha$ and $R(\Sigma, A_\alpha) = 0$, i.e. $\bigcap\{(0 : \mathfrak{M})_{A_\alpha} \mid \mathfrak{M} \in \Sigma_{A_\alpha}\} = 0$. But $(0 : \mathfrak{M})_{A_\alpha} = (0 : \mathfrak{M})_{A/T_\alpha}$, and therefore $\bigcap\{(0 : \mathfrak{M})_{A/T_\alpha} \mid \mathfrak{M} \in \Sigma_{A_\alpha}\} = 0$, i.e. $\bigcap\{(0 : \mathfrak{M})_A \mid \mathfrak{M} \in \Sigma_{A_\alpha}\} = T_\alpha$. By condition P.1, from the last equality we obtain $T_\alpha \supseteq \bigcap\{(0 : \mathfrak{M})_A \mid \mathfrak{M} \in \Sigma_A\}$. Consequently, $T_\alpha \supseteq R$. In view of the arbitrariness of the choice of the ideal T_α, $R \subseteq \bigcap T_\alpha$.

Corollary 3. A subdirect sum $A = \Sigma_s \oplus A_\alpha$ of Σ-semisimple rings A_α is a Σ-semisimple ring.

Indeed, if $A = \Sigma_s \oplus A_\alpha$, then in A there exist ideals T_α such that $\bigcap T_\alpha = 0$ and $A/T_\alpha \simeq A_\alpha$. By Proposition 4, $R(\Sigma, A) = 0$.

Let Σ be a general class of modules. Everywhere in what follows, by $\Omega(\Sigma)$ we shall denote the class of all rings A possessing faithful A-modules $\mathfrak{M}$ from the class Σ_A.

Corollary 4. Σ-semisimple rings are characterized by the fact that they are subdirect sums of rings from the class $\Omega(\Sigma)$.

Indeed, if A_α has a faithful A_α-module $\mathfrak{M}$ from Σ_A, then $(0 : \mathfrak{M})_{A_\alpha} = 0$, and therefore $R(\Sigma, A_\alpha) = 0$. By Corollary 3, $\Sigma_s \oplus A_\alpha = A$ will also be a Σ-semisimple ring. Conversely, if $R(\Sigma, A) = 0$, then A is a subdirect sum of the rings $A/(0 : \mathfrak{M})_A = \bar{A}$, where, by condition P.2 and Proposition 1, $\mathfrak{M} \in \Sigma_{\bar{A}}$ and $(0 : \mathfrak{M})_{\bar{A}} = (0 : \mathfrak{M})_A/(0 : \mathfrak{M})_A = 0$.

Proposition 5. A ring A will be Σ-semisimple if and only if there exists a faithful A-module that is a discrete direct sum of A-modules from the class Σ_A.

Indeed, if the ring A has a faithful A-module $\mathfrak{M} = \Sigma \oplus \mathfrak{M}_\alpha$—a discrete direct sum of A-modules from the class Σ_A, then $(0 : \mathfrak{M})_A = \bigcap(0 : \mathfrak{M}_\alpha)_A = 0$. Consequently, the ring A is a subdirect sum of rings from the class $\Omega(\Sigma)$, i.e. Σ-semisimple, by Corollary 4. Conversely, let A be Σ-semisimple, i.e.

$$R(\Sigma, A) = \bigcap\{(0 : \mathfrak{M})_A \mid \mathfrak{M} \in \Sigma_A\} = 0.$$

The set of ideals B of the ring A, having the form $(0 : \mathfrak{M})_A$, where $\mathfrak{M} \in \Sigma_A$, can be well ordered. Therefore one may assume that for each $(0 : \mathfrak{M})_A$ there is an index α from some well-ordered set Λ of indices such that $B_\alpha = (0 : \mathfrak{M})_A$. To each B_α we put in correspondence an A-module $\mathfrak{M} \in \Sigma_A$ such that $B_\alpha = (0 : \mathfrak{M})_A$, and denote this A-module by $\mathfrak{M}_\alpha$. Then the discrete direct sum $\mathfrak{M} = \Sigma \oplus \mathfrak{M}_\alpha$ is the required A-module. Indeed, all $\mathfrak{M}_\alpha \in \Sigma_A$, and

$$(0 : \mathfrak{M})_A = \bigcap_\alpha (0 : \mathfrak{M}_\alpha)_A = \bigcap\{(0 : \mathfrak{M})_A \mid \mathfrak{M} \in \Sigma_A\} = R(\Sigma, A) = 0.$$

Lemma 1. If B is a Σ-radical ideal of the ring A, then $B \subseteq R(\Sigma, A)$.

Indeed, if one assumes that $B \not\subseteq R(\Sigma, A)$, then there is an A-module $\mathfrak{M} \in \Sigma_A$ such that $B \not\subseteq (0 : \mathfrak{M})_A$. Then $B/(0 : \mathfrak{M})_B = \bar{B} = B/(0 : \mathfrak{M})_A \cap B \simeq B + (0 : \mathfrak{M})_A/(0 : \mathfrak{M})_A$ will be a nonzero ideal in a Σ-semisimple ring-

$\overline{A} = A/(0 : \mathfrak{M})_A$. By property P.3, $\Sigma_{\overline{B}} \neq \emptyset$. But then also $\Sigma_B \neq \emptyset$, by property P.1. We have obtained a contradiction to the Σ-radicality of the ideal B.

Corollary 5. The sum T of any set of Σ-radical ideals R_α of the ring A is a Σ-radical ideal.

Indeed, since for any α, R_α is a Σ-radical ideal also in the ring T, it follows, by Lemma 1, that $R(\Sigma, T) \supseteq R_\alpha$, and consequently $R(\Sigma, T) \supseteq T \supseteq R(\Sigma, T)$.

Proposition 6. If $\Sigma_{\overline{A}} = \emptyset$, where $\overline{A} = A/B$, and $\Sigma_B = \emptyset$, then $\Sigma_A = \emptyset$, i.e., an extension of a Σ-radical ring by means of a Σ-radical ring is itself Σ-radical.

Indeed, suppose that $\Sigma_A \neq \emptyset$ and $\mathfrak{M} \in \Sigma_A$. By Lemma 1, $B \subseteq R(\Sigma, A)$, and therefore $\mathfrak{M}B = 0$. According to condition P.2, $\Sigma_{\overline{A}} \neq \emptyset$, which contradicts the condition.

Proposition 7. The Σ-radical $R(\Sigma, A)$ of the ring A coincides with the sum T of all its Σ-radical ideals.

In view of Corollary 5 and Lemma 1, $R(\Sigma, A) \supseteq T$. By Proposition 6, the factor ring A/T contains no nonzero Σ-radical ideals. By property P.4, $\overline{A} = A/T$ is Σ-semisimple. By Proposition 4, $R(\Sigma, A) \subseteq T$.

Let us recall some definitions and results needed for what follows. Let s be an arbitrary property of rings. If a ring A has property s, one says that A is an s-ring. If an s-ring B is an ideal of the ring A, then B is called an s-ideal in A. We shall denote the sum of all s-ideals of the ring A by $s(A)$.

We shall say that the property s is **radical** (in the sense of Kurosh–Amitsur), or that the property s defines a radical s, if

I.1. A homomorphic image of an s-ring is an s-ring.
I.2. For every ring A, $s(A)$ is an s-ideal in A.
I.3. For every ring A, $s(A/s(A)) = 0$.

If the property s is radical, then s-rings are called s-radical; rings without nonzero s-ideals, i.e., rings A for which $s(A) = 0$, are called s-semisimple. Finally, the ideal $s(A)$ of the ring A is called the s-radical in A.

Let M be an arbitrary class of rings satisfying the condition

II.1. Every nonzero ideal of a ring from the class M is homomorphically mapped onto nonzero rings from M.

We shall call a ring A an S_M-ring if A is not homomorphically mapped onto nonzero rings from the class M. The property S_M turns out to be radical. The radical S_M is called the **upper radical** determined by the class M. Throughout what follows, by a radical we shall mean some general radical in the sense of Kurosh–Amitsur.

Theorem 1. Every Σ-radical is a radical. Conversely, if R is an arbitrary radical, then there exists such a general class Σ of modules that R coincides with the Σ-radical. In order that some class of rings M coincide with $\Omega(\Sigma)$

for a suitable general class Σ of modules, it is necessary and sufficient that the class M satisfy condition II.1 and that all S_M-semisimple rings be exhausted by subdirect sums of rings from M.

Proof. The first assertion follows from Propositions 2 and 7 and Corollary 1. Let now R be an arbitrary radical. Consider the class M of all R-semisimple rings and associate with each ring A the class Σ_A of all such nontrivial A-modules $\mathfrak{M}$ that the factor ring $\widetilde{A} = A/(0:\mathfrak{M})_A \in M$.

By Proposition 1 and the isomorphism $\overline{A}/(0:\mathfrak{M})_{\overline{A}} \cong A/(0:\mathfrak{M})_A$, we easily obtain that the class Σ of all Σ_A satisfies conditions P.1 and P.2. Let us note that, in this case, Propositions 2–4 and Corollaries 1–4 are true for the constructed class Σ, since in the proof of these assertions conditions P.3 and P.4 were not used.

We shall show that, for the constructed class Σ of modules over the ring A, the rings possessing faithful A-modules $\mathfrak{M} \in \Sigma_A$ are precisely the R-semisimple rings. Indeed, if the ring A possesses a faithful A-module $\mathfrak{M} \in \Sigma_A$, then A is R-semisimple by the construction of the classes Σ_A. Conversely, let the ring A be R-semisimple. Embed, in the usual way (see, for example, [6], p. 54), the ring A in a ring $\mathfrak{M}$ with identity and regard $\mathfrak{M}$ as an A-module. It is evident that $\mathfrak{M}$ is a faithful A-module. By the construction of the class Σ_A, $\mathfrak{M} \in \Sigma_A$.

Let the class Σ_A be faithful, i.e.

$$\bigcap\{(0:\mathfrak{M})_A \mid \mathfrak{M} \in \Sigma_A\} = 0.$$

The ring A has turned out to be a subdirect sum of R-semisimple rings

$$\bar{A} = A/(0:\mathfrak{M})_A.$$

Therefore the ring A itself is R-semisimple. By what has been proved, there exists a faithful A-module $\mathfrak{M} \in \Sigma_A$. If B is a nonzero ideal of the ring A, then $\mathfrak{M}B \neq 0$. But then B is mapped homomorphically onto the ring

$$\bar{B} = B/B \cap (0:\mathfrak{M})_A \simeq B + (0:\mathfrak{M})_A/(0:\mathfrak{M})_A$$

—a nonzero ideal of the R-semisimple ring

$$\bar{A} = A/(0:\mathfrak{M})_A.$$

Consequently, $\bar{B}$ is mapped homomorphically onto nonzero R-semisimple rings, i.e., by what has been proved, onto rings possessing faithful modules from the class Σ. By Corollary 2, $\Sigma_{\bar{B}} \neq \emptyset$, and hence $\Sigma_B \neq \emptyset$. Thus condition P.3 is fulfilled. Conversely, suppose that for every nonzero ideal B of the ring A, $\Sigma_B \neq \emptyset$. This means that B is mapped homomorphically onto rings possessing faithful modules from the class Σ, i.e., onto R-semisimple rings. Consequently, B is not an R-radical ring. Therefore the ring A is R-semisimple. But then the class Σ_A is faithful, since, by what has been proved, there exists a faithful A-module from the class Σ_A. Thus condition P.4 is also fulfilled.

From everything proved above and from Corollary 2 we obtain that Σ is a general class of modules $M = \mathfrak{L}(\Sigma)$, and the ring A is Σ-radical if and only if it is R-radical. Thus the second assertion is also proved.

Finally, let M be an arbitrary class of rings satisfying condition P.1, and suppose that the S_M-semisimple rings are exhausted by subdirect sums of rings from the class M. Quite analogously to the construction of the class Σ carried out for the class of semisimple rings, we construct such a general class Σ of modules that $M = \mathfrak{L}(\Sigma)$. The converse is evident by condition P.3 and Corollary 4. The theorem is proved.

Recall that a radical R is called hereditary if every ideal of an R-radical ring is R-radical. Amitsur, in [2], showed that the radical R is hereditary (in the class of associative rings) if and only if, for every ring A and every ideal B of A, the equality

$$R(B) = B \cap R(A)$$

holds. But in that case, quite analogously to the proof of Theorem 1, we obtain that the following is true.

Theorem 2. Let Σ be a class of modules for which conditions P.1, P.2 and

P.5. For every ring A and every ideal B of A the equality

$$\operatorname{Ker} \Sigma_B = B \cap \operatorname{Ker} \Sigma_A$$

holds.

Then Σ is a general class of modules, and the Σ-radical is a hereditary radical. Conversely, if Σ is such a general class of modules that the Σ-radical is a hereditary radical, then condition P.5 is fulfilled. For every hereditary radical R there exists such a general class of Σ-modules, satisfying condition P.5, that R coincides with the Σ-radical.

Institute of Physics and Mathematics
Academy of Sciences of the MSSR

Received
22 II 1964

References

1. A. G. Kurosh, *Mat. sborn.*, **33** (75), 1, 13 (1953).

2. S. A. Amitsur, *Am. J. Math.*, **76**, 100 (1954).

3. N. Jacobson, *Structure of Rings*, IL, 1961.

4. V. A. Andrunakievich, *Sibirsk. matem. zhurn.*, **2**, No. 6, 801 (1961).

5. V. A. Andrunakievich, Yu. M. Ryabukhin, *DAN*, **147**, No. 6, 1274 (1962).

6. A. G. Kurosh, *Lectures on General Algebra*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.