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Abstract

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ON THE STATIONARY APPROACH IN ABSTRACT SCATTERING THEORY

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1. In the present note, within the framework of the abstract theory of operators, a justification is given for the so-called stationary version of scattering theory. Under the assumption that the perturbation is nuclear, convenient representations are given (see formulas (3)–(5)) for the wave operators* and scattering operators in terms of boundary values of resolvents; also justified is one practically convenient method for computing the S-matrix. At the same time, a new, purely "stationary," proof is obtained of the Rosenblum–Kato theorem ($^{4-6}$) on the existence of wave operators. A methodological advantage of the method proposed here consists, among other things, in the absence of a limiting passage from finite-dimensional perturbations. This makes it possible to transfer the results obtained to the case of more general perturbations, which, in particular, makes them applicable in the three-dimensional problem of quantum scattering. Let us also note here that relation (3) is the "correct" form of the well-known Lippmann–Schwinger equation (^{7}).

2. Let us first record several auxiliary propositions. Let A be a self-adjoint operator in a Hilbert space $\mathfrak{H}$, F_λ its resolution of the identity, Γ_z its resolvent, and D some operator of class** $\mathfrak{S}_2$. Then:

1°. For any element $f \in \mathfrak{H}$, for almost all (a.e.) λ there exists the strong derivative $d(DF_\lambda f)/d\lambda$.

2°. For a.e. λ there exist the strong limits of the element $D\Gamma_z f$, when $z \to \lambda \pm i0$, i.e., when $z \to \lambda$ along a path not tangent to the real axis in the upper (lower) half-plane.

3°. For a.e. λ there exists the strong derivative

$$d(D^* F_\lambda D)/d\lambda \equiv K_\lambda \in \mathfrak{S}_1.$$

4°. For a.e. λ, as $z \to \lambda \pm i0$, there exist the strong limits $M_\lambda^{(\pm)}$ of the operators $D^*\Gamma_{zD}$, with $M_\lambda^{(\pm)} \in \mathfrak{S}_2$ and

$$M_\lambda^{(+)} - M_\lambda^{(-)} = -2\pi i K_\lambda \in \mathfrak{S}_1.$$

Propositions 1° and 3° are essentially known. They follow easily from I. M. Gelfand' s theorem (^{8}) on the differentiability of functionals with strongly bounded variation. Propositions 2° and 4° are derived from 1° and 3° with the aid of the theorem on boundary values of Cauchy—Stieltjes integrals. Let us note that the exceptional null sets of values λ in 1° and 2° depend, generally speaking, on the element f.

3. Everywhere in what follows, H_0, H are self-adjoint operators in $\mathfrak{H}$ with common domain of definition $\mathfrak{D}$; E^0_λ, E_λ and R^0_z, R_z are their corresponding resolutions of the identity and resolvents; $V = H - H_0$; P_0, P are the projectors onto the absolutely continuous subspaces $\mathfrak{G}_0, \mathfrak{G}$ of the operators H_0, H

Accordingly. Put also $Q^0_z = E + VR^0_z$, $Q_z = E - VR_z$, and note that $Q^0_z = Q_z^{-1}$, and

$$R^0_z - R_z = Q^{0*}_z (R_z - R_{\bar z}) Q^0_z. \tag{1}$$

If $V \in \mathfrak{S}_1$, then, by virtue of 2°, a.e. there exist strong limits $Q^0_{\lambda \pm i0} f$ of the elements $Q^0_z f$. Similarly for $Q_z f$. In addition, by virtue of 1°—3°, for any $f, g \in \mathfrak{H}$, a.e. in λ and a.e. in μ there exist the derivatives $d(E_\lambda Q^0_{\mu \pm i0} f, g)/d\lambda$ and $d(E_\lambda Q^0_{\mu \pm i0} f, Q^0_{\mu \pm i0} g)/d\lambda$. Finally, passing to the limit in (1), with the aid of 1°—4° one can find that a.e.

$$\left[d(E_\lambda Q^0_{\mu \pm i0} f, Q^0_{\mu \pm i0} g)/d\lambda\right]_{\mu=\lambda} = d(E^0_\lambda f, g)/d\lambda. \tag{2}$$

From (2) it follows immediately that

$$\int_{-\infty}^{+\infty} \left| \left[d(E_\lambda Q^0_{\mu \pm i0} f, g)/d\lambda\right]_{\mu=\lambda} \right| d\lambda \leqslant \|P_0 f\| \, \|Pg\|.$$

We can now introduce bounded operators $W_\pm$ by means of the relation

$$(W_\pm f, g) = \int_{-\infty}^{+\infty} \left[d(E_\lambda Q^0_{\mu \pm i0} f, g)/d\lambda\right]_{\mu=\lambda} d\lambda. \tag{3}$$

Theorem 1. *Under the condition $V \in \mathfrak{S}_1$, the wave operators $W_\pm(H, H_0)$ coincide with the operators $W_\pm$ defined by the relations (3). Along with (3), for $W_\pm$ the representation*

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* For the definition of wave operators, the scattering operator, and other notions of abstract scattering theory, see, for example, (^{1}) or (^{2}). An abstract definition of the S-matrix is given in (^{3}).

** By $\mathfrak{S}_2$ we denote the class of Hilbert—Schmidt operators, and by $\mathfrak{S}_1$ the class of nuclear operators.

$$(W_{\pm}f, g) = \int_{-\infty}^{+\infty} \left[d(E_{\lambda}^{0}f, Q_{\mu\pm i0}g)/d\lambda\right]_{\mu=\lambda} d\lambda \tag{4}$$

is valid.

Let us note that the wave operators exist by the Rosenblum–Kato theorem. They map $\mathfrak{G}_0$ isometrically onto $\mathfrak{G}$ and implement a unitary equivalence of the absolutely continuous parts of H_0 and H. These properties of the operators $W_{\pm}$ can also be derived directly from relations (2)–(4). Taking into account the connection between the definition (3) and the usual nonstationary definition of the wave operators, we arrive at a new (“stationary”) proof of the existence of wave operators for nuclear perturbations. Let us point out in this connection that, in the case of a one-dimensional perturbation, a stationary proof was given earlier by T. Kato (^{5}). Formula (3) was indicated in a somewhat different form by one of the authors (see formula (7) in (^{9})).

The scattering operator $S = W_{+}^{*}W_{-}$ admits representations of the same type as (3), (4). Namely, for any $f, g \in \mathfrak{H}$,

$$(Sf, g) = \int_{-\infty}^{+\infty} \left[d(E_{\lambda}Q_{\mu-i0}^{0}f, Q_{\lambda+i0}^{0}g)/d\lambda\right]_{\mu=\lambda} d\lambda,$$

$$(Sf, g) = \int_{-\infty}^{+\infty} \left[d(E_{\lambda}^{0}Q_{\mu+i0}Q_{\mu-i0}^{0}f, g)/d\lambda\right]_{\mu=\lambda} d\lambda. \tag{5}$$

4. Let us now recall the abstract definition of the S-matrix given in (^{3}). Let $\mathfrak{G}_0$ be decomposed into a continuous direct sum (^{10}) of Hilbert spaces

$$\mathfrak{G}_0 = \int_{\Lambda} \oplus \mathfrak{H}_{\lambda}\, d\lambda \tag{6}$$

so that the part of the operator H_0 in $\mathfrak{G}_0$ becomes the operator of multiplication by λ. Here Λ is the spectrum of H_0 in $\mathfrak{G}_0$. The operator S is unitary in $\mathfrak{G}_0$ and commutes with H_0, and therefore in the decomposition (6) there corresponds to it a measurable family of unitary operators S_{λ} in $\mathfrak{H}_{\lambda}$. The operator S_{λ}, defined for a.e. $\lambda \in \Lambda$,

is called the S-matrix (the scattering suboperator). The bilinear form S_{λ} coincides with the integrand in formula (5). We describe another method for computing the S-matrix.

Introduce into consideration the operator $T_z = P_0 Q_{zVP}0$. In the expansion (6), the operator T_z corresponds to the “kernel” $T_z(\lambda, \mu)$ in the following sense. $T_z(\lambda, \mu)$ is a measurable family of nuclear operators, defined for a.e. $\lambda \in \Lambda$ and a.e. $\mu \in \Lambda$, and acting from $\mathfrak{h}_{\mu}$ into $\mathfrak{h}_{\lambda}$. Moreover, if $f(\lambda) \in \mathfrak{h}_{\lambda}$ is the representation of the element $f \in \mathfrak{G}_0$ in the expansion (6), then

$$(T_z f)(\lambda) = \int_\Lambda T_z(\lambda, \mu) f(\mu)\, d\mu. \tag{7}$$

Denote by I_λ the identity operator in $\mathfrak{h}_\lambda$. The following holds.

Theorem 2. *For a.e. real ν, a.e. $\lambda \in \Lambda$, and a.e. $\mu \in \Lambda$, the operators $T_z(\lambda, \mu)$ converge in the nuclear norm as $z \to \nu + i0$. The limiting operator $T_{\nu+i0}(\lambda, \mu)$ is related to the scattering suboperator S_λ by the relation*

$$S_\lambda = I_\lambda - 2\pi i T_{\lambda+i0}(\lambda, \lambda) \quad (\text{for a.e. } \lambda \in \Lambda).$$

The results of §§ 3, 4 are easily carried over to the case of a pair of unitary operators, first considered in (3).

5. We now give one generalization of the preceding results, which makes it possible to use them directly in applications. By the same method as Theorem 1, one proves the following.

Theorem 3. *Let $|V|^{1/2}(H_0 - iE)^{-n} \in \mathfrak{S}_2$, $|V|^{1/2}(H - iE)^{-n} \in \mathfrak{S}_2$ $(n = 1, 2, \ldots)$. Then there exist the wave operators $W_\pm(H, H_0)$, $W_\pm(H_0, H)$.*

We note that under the hypotheses of Theorem 3 the representation (3) remains valid for arbitrary $g \in \mathfrak{H}$ and for a dense set of $f \in \mathfrak{H}$. For $n = 1$, Theorem 3 contains the criterion for the existence of wave operators due to S. T. Kuroda (11), and, in turn, is contained in the criterion obtained by M. Sh. Birman and M. G. Kreĭn in (3). For $n > 1$, Theorem 3 gives a new criterion for the existence of wave operators, containing the results of I. V. Stankevich (12). It is interesting that for $n > 1$ the result obtained does not follow from the general criterion for the existence of wave operators (13, 2), formulated in terms of functions of operators.

Under the hypotheses of Theorem 3 the operator T_z is in any case defined on elements $f \in \mathfrak{D}$. It is easy to show that now also the operator T_z admits the integral representation (7) with kernel $T_z(\lambda, \mu) \in \mathfrak{S}_1$.

Theorem 4. *Under the hypotheses of Theorem 3, the assertions of Theorem 2 are valid.*

In conclusion we note that the rule for computing S_λ given by Theorems 2 and 4 is, from a formal point of view, known to physicists. For the three-dimensional problem of quantum scattering this rule was justified by L. D. Faddeev (14). However, even in application to this problem, Theorem 4 gives new information, since in (14) the limiting passage is understood in a different sense and is justified in a special (momentum) representation.

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Note: Figure translations are in progress. See original paper for figures.