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Abstract

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MATHEMATICS

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GENERAL BOUNDARY-VALUE PROBLEMS WITH DISCONTINUOUS BOUNDARY CONDITIONS

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1. The case of an elliptic differential equation of order $2m$

In a domain $G \subset R^n$ with smooth boundary Γ, a general elliptic differential equation of order $2m$ is given:

$$A(x, D)u(x) = f(x). \tag{1}$$

Let Γ be divided into two parts Γ^+ and Γ^-, with $\gamma = \Gamma^+ \cap \Gamma^-$ a smooth $(n-2)$-dimensional surface. On Γ^+ and Γ^- boundary conditions are prescribed:

$$B_{1j}(x, D)u|_{\Gamma^+} = \varphi_{1j}(x'), \qquad B_{2j}(x, D)u|_{\Gamma^-} = \varphi_{2j}(x') \quad (1 \le j \le m). \tag{2}$$

It is assumed that B_{1j} on Γ^+ and B_{2j} on Γ^- satisfy the Shapiro-Lopatinskii condition, or, for brevity, we shall say: the natural conditions. Fix a point $x_0 \in \gamma$ and introduce in its neighborhood a local coordinate system (x', x_n), where the axis x_n is normal to Γ, and x' lies in the tangent plane to Γ. By (ξ', ξ_n) we shall always denote the variables dual to (x', x_n). Let

$$A_0(x_0, \xi) = A_+(x_0, \xi)A_-(x_0, \xi)$$

be the factorization of the principal part $A_0(x, \xi)$ of the polynomial $A(x, \xi)$ with respect to ξ_n. Denote

$$b_{ijk}(x_0, \xi') = \int_C \frac{B_{ij}^{(0)}(x_0, \xi)\xi_n^{k-1}}{A_+(x_0, \xi)}\, d\xi_n \quad (i = 1, 2;\ 1 \le j, k \le m),$$

where C is a contour in the complex ξ_n-plane enclosing the zeros of $A_+(x_0, \xi)$; $B_{ij}^{(0)}(x_0, \xi)$ is the principal part of $B_{ij}(x_0, \xi)$, ord $B_{ij}^{(0)}(x_0, \xi) = m_{ij}$.

Introduce the matrices

$$B_i = \|b_{ijk}(x_0, \xi')\|_{j,k=1}^{m} \quad (i = 1, 2).$$

The matrix $B_1 B_2^{-1}$ can be represented in the form

$$B_1 B_2^{-1} = D_1 B_3 D_2^{-1}, \tag{3}$$

where B_3 is a matrix of zero order of homogeneity;

$$D_1 = \|\xi_-^{m_{1j}} \delta_{jk}\|, \qquad D_2 = \|\xi_+^{m_{2j}} \delta_{jk}\|,$$

where

$$\xi_+ = \xi_{n-1} + i|\xi''|, \qquad \xi_- = \xi_{n-1} - i|\xi''|, \qquad \xi' = (\xi'', \xi_{n-1}).$$

Here x_{n-1} is normal to γ, and x'' lies in the tangent plane to γ.

It is assumed that at every point $x_0 \in \gamma$ the condition of proper ellipticity is fulfilled, i.e. B_3 admits a factorization of the form

$$B_3 = B_3^+ D_+ D_-^{-1} B_3^-, \tag{4}$$

where B_3^+, B_3^- are matrices of zero order of homogeneity in ξ', depending smoothly on x' and ξ' ($|\xi'| = 1$), with B_3^+ (B_3^-) admitting analytic continuation in ξ_{n-1} to the upper (lower) half-plane and, moreover,

$$\det B_3^+(x_0, \xi') \neq 0 \quad \text{for} \quad \operatorname{Im} \xi_{n-1} \geq 0,\ |\xi'| > 0,$$

and

$$\det B_3^-(x_0, \xi') \neq 0 \quad \text{for} \quad \operatorname{Im} \xi_{n-1} \leq 0,\ |\xi'| > 0.$$

The matrices $D_\pm$ have the form

$$D_\pm = \|\delta_{jk} \xi_\pm^{\varkappa_j(x_0)}\| \quad (x_0 \in \gamma)$$

(see ([3])).

First consider the special case in which all $\varkappa_j(x)$ do not depend on x and are equal to one another. Denote by $\dot{H}_{\lambda,N}(G)$ the space of functions $u(x)$ with norm

$$\|u\|_{\lambda,N}^2 = \sum_{k=0}^{N} \|\alpha_k(x) u(x)\|_{\lambda+k}^2,$$

where $N \geq 0$ is an integer, $N \geq s_0 - \varkappa,\ s_0 =$

$\max(2m, m_{1i} + 1/2, m_{2j} + 1/2),\ a_k(x) \in C^\infty,\ a_k(x) = O(r^k),\ r$ is the distance to $\gamma,\ a_k(x) > 0,\ x \in G.$

Theorem 1. The problem (1), (2) is normally solvable in the space $H_{\chi+\delta,N}(G)$, where $0 < \delta < 1$, if $f \in H_{\chi+\delta-2m,N}(G)$, $\varphi_{1j} \in H_{\chi+\delta-m_{1j}-1/2,N}(\Gamma^+)$, $\varphi_{2j} \in H_{\chi+\delta-m_{2j}-1/2,N}(\Gamma^-)$. In this case the estimate holds

$$\|u\|_{\chi+\delta,N} \leq C\left(\|f\|_{\chi+\delta-2m,N} + \sum_{j,i}\|\varphi_{ji}\|_{\chi+\delta-m_{ij}-1/2,N} + \|u\|_{\chi+\delta-1,N}\right). \quad (5)$$

We note that the space $H_{\chi+\delta,N}(G)$ consists of functions having smoothness of order $\chi+\delta+N$ outside γ and smoothness only of order $\chi+\delta$ in a neighborhood of γ. We emphasize that in the space $H_{\chi+\beta,N}(G)$, for $\beta \geq 1$ or $\beta \leq 0$, the problem (1), (2) is no longer normally solvable (see (1)).

Remark. In the case when $\chi(x)$ depends on $x \in \gamma$, there is also a theorem on the normal solvability of the problem (1), (2) in the spaces $H_{\chi(x)+1/2,N}$, $N \geq s_0 - \min\chi(x)$, $x \in G+\Gamma$, defined by means of a partition of unity in $G+\Gamma$, analogously to ([1]).

It is of interest to study in greater detail the behavior of the function $u(x)$ in a neighborhood of γ. It turns out that even in the more general case of constant, but different χ_j, $u(x)$ can be represented in the form

$$u(x) = v(x) + \sum_{j=1}^{m}\sum_{i=1}^{k_i}\int_\gamma G_{ij}(x, x-y'')\, g_{ij}(y'')\, dy'' = v(x) + \sum_{j=1}^{m}\sum_{i=1}^{k_i} G_{ij}g_{ij}, \quad (6)$$

where $v(x) \in H^s(G)$, and the kernels $G_{ij}(x,z)$ are expressed in terms of the coefficients of the operators A, B_{1j}, B_{2j}. The Fourier transforms with respect to z, $F_{zG_{ij}}$, have a principal part of homogeneity order $-\chi_j - i - 1$. Here it is assumed that $s - \sigma_j \neq l$ (cf. ([2],[4])), l is any integer, $i = 1, 2, \dots, m$; $\sigma_j = \operatorname{Re}\chi_j$; $s - \sigma_j - 1/2 = k_j + r_j$, where k_j is an integer, $|r_j| < 1/2$.

Theorem 2. If in (1) and (2) one substitutes the expression (6) in place of $u(x)$, then the resulting problem of finding $v(x)$ and $g_{ij}(x'')$ ($x \in G$, $x'' \in \gamma$) is normally solvable in the spaces: $v(x) \in H^s(G)$, $g_{ij}(x'') \in H^{s-\chi_j+i-1}(\gamma)$, $f \in H^{s-2m}(G)$, $\varphi_{1j} \in H^{s-m_{1j}-1/2}(\Gamma^+)$, $\varphi_{2j} \in H^{s-m_{2j}-1/2}(\Gamma^-)$ ($s \geq s_0$). The corresponding a priori estimate, analogous to (5), holds.

Theorem 2 makes it possible to resolve the question of the smoothness of $u(x)$ near γ. The principal singularities of the solution $u(x)$ are contained in the potentials on γ, namely: $G_{ij}g_{ij}$, for smooth g_{ij}, has a singularity of order r^{χ_j+i-1}, possibly with a logarithmic factor. Therefore, in order that $u(x)$ be a smooth function in $G+\Gamma$, it is necessary that all $g_{ij}(x'') = 0$ on γ, and this gives an infinite number of conditions on the right-hand sides f and φ, on which $g_{ij}(x'')$ depends. We note that from (6) one can obtain the asymptotic behavior of the solution $u(x)$ in a neighborhood of γ.

2. Systems of paired equations. In the study of general boundary value problems an important role is played by the study of systems of paired equations on Γ, i.e. systems of the form

$$\sum_{j=1}^{p} B_{kj}^{(1)} u_j = f_{k1}(x'), \qquad x' \in \Gamma^+, \qquad k = 1, \dots, p; \tag{7}$$

$$\sum_{j=1}^{p} B_{kj}^{(2)} u_j = f_{k2}(x'), \qquad x' \in \Gamma^-, \qquad k = 1, \dots, p, \tag{8}$$

where $B_{kj}^{(1)}$ and $B_{kj}^{(2)}$ are singular operators on all of Γ, and $u_j(x')$ are unknown functions on Γ. If the symbols $\widetilde{B}_1$ and $\widetilde{B}_2$ of the operators $B^{(1)}$ and $B^{(2)}$ are elliptic on Γ and the matrix $\widetilde{B}_1 \widetilde{B}_2^{-1}$ admits on γ a factorization of the form (4), then the system (7), (8) is normally solvable in spaces analogous to those introduced in Theorem 5 in ([3]).

Instead of (7), (8) one may consider a more general system when, for example, r_1 of the functions $u_j(x') = 0$ on Γ^-, and r_2 of the functions $u_j(x') = 0$ on Γ^+. Then (7) consists of $p - r_2$ equations, and (8) of $p - r_1$ equations. In this case as well an analogous theorem on normal solvability holds.

3. General problems with discontinuous boundary conditions. Let an equation, generally speaking of variable order $\alpha(x)$, be given in G:

$$L_{\alpha(x)} u(x) = f(x), \tag{9}$$

where $L_{\alpha(x)}$ has the same form as in ([3]). Let $L_{\alpha(x)} = L^- L^+ + T_N$ be a factorization of $L_{\alpha(x)}$, where L^+, generally speaking, does not satisfy condition c) of ([1,3]), and has nonintegral order $\varkappa(x)$. Then, as was shown in ([3]), for (9) one may impose any finite number of boundary conditions, with the order of growth of the desired solutions as they approach Γ increasing by one when the number of boundary conditions is increased by one. Therefore one may pose the question of finding a solution of (9) satisfying one number of boundary conditions on Γ^+ and another number on Γ^-. It is then natural to expect that the solutions have one order of growth when approaching Γ^+ and another when approaching Γ^-. Let the boundary conditions have the form:

$$B_{j1} u\big|_{\Gamma^+} = \varphi_{j1}(x'), \qquad 1 \le j \le M_+; \tag{10}$$

$$B_{j2} u\big|_{\Gamma^-} = \varphi_{j2}(x'), \qquad 1 \le j \le M_-, \tag{11}$$

where $M_+ \ge M_-$, and B_{j1} and B_{j2} satisfy on Γ^+ and Γ^- the conditions of Theorem 3 from ([3]); $\operatorname{ord} B_{j1} = a_{j1}(x)$, $\operatorname{ord} B_{j2} = a_{j2}(x)$.

The solution of this problem, roughly speaking, can be found in the form

$$u(x) = \theta^+ R^+ \left(v_+(x) + \sum_{k=1}^{M_+} c_k(x') \delta^{(k-1)}(\Gamma) \right), \tag{12}$$

where $L^+ \cdot R^+ = I + T_N$ (see ([3])); θ^+ is the characteristic function of the domain $G+\Gamma$; $v_+(x)$ is a function in G; $c_k(x')$ are functions on Γ, with $c_k(x') = 0$ for $k > M_-$, $x' \in \Gamma^-$; $\delta^{(k)}(\Gamma)$ is the derivative of the δ-function in the normal direction to Γ. Substituting (12) into (9), (10), (11), we obtain equations for finding $v_+(x)$ and $c_k(x')$. For $c_k(x')$ this yields a system of paired equations of the same type as in the preceding section. It is assumed here that the corresponding matrix $\widetilde{B}_1 \widetilde{B}_2^{-1}$ admits a factorization of the form (4) on γ. Denote by H the space of functions $u(x)$ that admit the representation (12), where $v_+(x) \in H^{l(x)-\varkappa(x)}(G)$, and $c(x') = (c_1(x'), \dots, c_{M_+}(x'))$ belongs to the space H_1 of the same type as in Theorem 5 in ([3]) (see the preceding section). For brevity we do not describe H_1 in detail here.

Theorem 3. The problem (9), (10), (11) is normally solvable for $u(x) \in H$,

$$f(x) \in H^{l(x)-\alpha(x)}(G), \qquad \varphi_{j1}(x') \in H^{l(x)-a_{j1}(x)-1/2}(\Gamma^+), \qquad \varphi_{j2}(x') \in H^{l(x)-a_{j2}(x)-1/2}(\Gamma^+).$$

An estimate analogous to (5) holds.

Remark 1. One may consider an even more general problem when, in addition to conditions (10) and (11) on Γ^+ and Γ^-, a certain number of conjugation conditions are prescribed on γ:

$$C_{j1} u\big|_{\gamma^+} - C_{j2} u\big|_{\gamma^-} = \Psi_j(x''), \qquad 1 \le j \le M_1,$$

where C_{j1}, C_{j2} are operators of the type B_{j1}, and $\big|_{\gamma^+}$ means that the restriction to γ is carried out in two steps: first from G to Γ^+, and then along Γ^+ to γ. The symbol $\big|_{\gamma^-}$ has an analogous meaning. The corresponding theorem on normal solvability has been proved.

Remark 2. Analogously to item 5 in ([3]), instead of equation (9) one may consider an equation of the form

$$L_\alpha \left(u + \sum_{k=1}^{M_+} G_k g_k \right) = f(x), \qquad x \in G, \tag{13}$$

where $g_k(x') = 0$ for $k > M_-$, $x' \in \Gamma$. Under the fulfillment of natural conditions, this problem is also normally solvable in the corresponding spaces.

4. We can now investigate such a general problem. Let the boundary of the domain Γ be divided into p parts:

$$\Gamma = \bigcup_{k=1}^{p} \Gamma_k,$$

where $\gamma_{ij} = \Gamma_i \cap \Gamma_j$ is either empty or is a closed $(n-2)$-dimensional manifold without singular points. On each part Γ_j, an arbitrary number of boundary conditions of the form (10) is prescribed, or an arbitrary number of additional potentials with densities equal to zero outside Γ_j is added to (9). Obviously, both the boundary conditions and the potentials must satisfy the natural conditions on Γ. Then, applying the results of the preceding item, we conclude that *in the corresponding space H of functions $u(x)$ (having the corresponding behavior near the various Γ_j) the indicated problem is normally solvable, if the right-hand sides belong to the usual spaces of type H^l.*

For general equations of the form (9), for which $\varkappa(x)$ is, generally speaking, variable, the following question is of particular interest: how should one divide the boundary Γ into parts Γ_i, and how many boundary conditions should be prescribed on each part, or how many additional potentials with densities concentrated on Γ_i should be added to (9), in order to obtain a normally solvable problem in the class of solutions continuous up to the boundary Γ and smooth inside G. At the same time it is natural to require that the number of boundary conditions on each piece Γ_i be maximal, and that the number of additional potentials in (9) be minimal. For definiteness, consider the case when the surfaces $\operatorname{Re}\varkappa(x') = k$, $x' \in \Gamma$, where k is an integer, are surfaces of type γ_{ij}. Then it turns out that one should take

$$\Gamma_k = \{x' \in \Gamma : \ k \leqslant \operatorname{Re}\varkappa(x') \leqslant k+1\}$$

and, for $k > 0$, prescribe on Γ_k k boundary conditions of the form (10), while for $k < 0$ add to equation (9) $|k|$ additional potentials. If on Γ_k they satisfy the natural conditions, then in the corresponding space H of functions $u(x)$ the problem is normally solvable; moreover $u(x)$ has the required smoothness properties, except, possibly, for the transition surfaces $\gamma_k = \Gamma_{k-1} \cap \Gamma_k$, where $u(x)$ may have certain singularities.

To ensure that $u(x) \in C(\overline{G})$, it is necessary additionally to add to equation (9) the corresponding number of potentials taken over γ_k. The proof of this theorem is carried out analogously to the proof of Theorems 3 and 4 in ([3]); here the spaces of the right-hand sides are the usual spaces of type H^l, while the spaces H of solutions $u(x)$ are constructed in a special way and, as is proved, are contained in $C(\overline{G})$.

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Note: Figure translations are in progress. See original paper for figures.