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Abstract

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ON THE POSSIBILITY OF EXTENDING THE UNIQUENESS PROPERTY OF ANALYTIC FUNCTIONS TO NOWHERE DENSE CLOSED SETS

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As is known, from the uniform convergence of a sequence of analytic functions inside a domain there follows the uniqueness property of the limiting function. Namely, if the limiting function vanishes on some portion of this open set, then it is identically equal to zero.

Let E be a nowhere dense bounded closed set. Denote by A_E the set of functions analytic on E, and by A_E^∞ the closure of this set in the uniform metric. It is known that if E is a "fat" set, then A_E^∞ coincides with the set of all functions continuous on E (^{1}). In the contrary case A_E^∞ does not coincide with the indicated set. This fact gives grounds for posing the following problem. Does there exist a nowhere dense closed set E, sufficiently "massive" in some sense, for which, in the case of a domain, A_E^∞ possesses an analogous uniqueness property? It turns out that much stronger assertions are valid.

For the exposition we introduce some notation. By $\mathcal{L}_p(E)$, $p \geq 1$, we denote the Banach space whose elements are complex measurable functions $f(z)$ of the complex variable z, defined on E, with finite norm

$$\|f\|_p = \left\{ \iint_E |f(z)|^p \, dx \, dy \right\}^{1/p}, \qquad z = x + iy.$$

By A_E^p we denote the closure of the set A_E in the space $\mathcal{L}_p(E)$, $p \geq 1$.

Theorem 1. *There exists a nowhere dense bounded closed set E_0 such that from the equivalence to zero of each function of the set $A_{E_0}^2$ on some portion of this set there follows its equivalence to zero on all of E_0.*

From this theorem there obviously follows

Corollary 1. *The indicated uniqueness property also holds in any class $A_{E_0}^p$, $2 \leq p \leq \infty$.*

On nowhere dense closed sets it is possible to extend a deeper uniqueness property of analytic functions.

Theorem 2. *If the sequences of numbers* $\{r_m\}$ *and* $\{\eta_m\}$ *are such that*

$$r_m \downarrow 0, \quad \eta_m \downarrow 0, \quad r_m \log \frac{1}{\eta_m} \to \infty \quad \text{as } m \to \infty,$$

then there exists a nowhere dense bounded closed set E_0^* *such that from the conditions*

$$f(z) \in A^2_{E_0^*}, \qquad \iint_{K_m^0 \cap E_0^*} |f(z)|^2 \, dx \, dy < \eta_m, \qquad m = 1, 2, 3, \dots,$$

where $K_m^0 :\ (|z - z_0| < r_m)$ and r_0 belongs to a certain measurable set $F_0 \subset E_0^*$, $m(E_0^* \setminus F) = 0$, there follows the equivalence to zero of $f(z)$ on E_0^*.

It is easy to show that the order of the sequence $\{n_m\}$ is finite, and the result in this direction cannot be improved. In addition, let us note that it is impossible to take the whole set E_0^* as F_0. This is not hard to verify by the example of the function $\exp\left(\frac{1}{z - z_0}\right)$, which excludes such a possibility even for domains.

Here, too, on the basis of this theorem we note the obvious corollary.

Corollary 2. *Under the conditions of Theorem 2, the assertion of this theorem is true for any* $A^p_{E_0^*}$, $2 \leqslant p \leqslant \infty$.

Let us indicate how the sets E_0 and E_0^* are constructed.

Consider the closed unit square Δ. Take a sequence of odd numbers

$$1 = n_0 < n_1 < \dots < n_k < \cdots .$$

Divide Δ, by straight lines parallel to the sides, into n_1^2 equal squares. Denote the resulting central open square by $\Delta_1^{(1)}$, and the remaining squares by $\delta_i^{(1)}$, $1 \leqslant i < n_1^2$. Each of the squares $\delta_i^{(1)}$, in turn, is divided by straight lines parallel to the sides into n_2^2 equal squares. Denote the central open square obtained in $\delta_i^{(1)}$ by $\Delta_i^{(2)}$, $1 \leqslant i < n_1^2$, and the remaining squares of the second division by $\delta_p^{(2)}$.

Suppose that the squares $\delta_i^{(k-1)}$, $1 \leqslant i < N_{k-1}^2$, $N_{k-1} = n_0 n_1 \cdots n_{k-1}$, have already been constructed. Divide each of these squares by straight lines parallel to the sides into n_k^2 equal squares. The open squares central in $\delta_i^{(k-1)}$ we denote by $\Delta_i^{(k)}$, $1 \leqslant i < N_{k-1}^2$. The remaining squares of the k-th division we denote by $\delta_p^{(k)}$, $1 \leqslant p < N_k^2$. Continue this process indefinitely.

Define the set E_0^* (or E_0) by putting

$$E_0^* = \Delta \setminus \left(\bigcup_{k=1}^{\infty} \bigcup_i \Delta_i^{(k)} \right).$$

Proof of Theorem 1. Let $f(z)$ be an arbitrary function from the space $\mathcal{L}_2(E_0)$, and let $\{R_\varepsilon(z)\}$ be a family of rational functions whose poles do not lie on E_0 and

$$\frac{B}{2} < \|R_\varepsilon\|_2^2 < 2B, \qquad B = \|f\|_2^2, \qquad \lim_{\varepsilon\to+0} \|f - R_\varepsilon\|_2 = 0.$$

We may assume that outside Δ the functions $R_\varepsilon(z)$ have no finite poles, and that the centers of all squares $\Delta_i^{(k)}$—$a_{k,i}$, $k \leqslant M(\varepsilon)$—are poles for the function $R(z) = R_\varepsilon(z)$. By $M = M(\varepsilon)$ we denote the largest number k for which the poles of this function fall into the squares $\Delta_i^{(k)}$, $1 \leqslant i < N_{k-1}^2$.

Write the expansion of $R_\varepsilon(z)$ into partial fractions:

$$R_\varepsilon(z) = P_\varepsilon(z) + \sum_{k=1}^{M} \sum_i \sum_{j\geqslant 1} \frac{C_j(k,i)}{(z - a_{k,i})^j},$$

where $P_\varepsilon(z)$ are polynomials.

Denote

$$R_{k,i}(z) = R_{k,i}^{(\varepsilon)}(z) = \sum_{j\geqslant 1} \frac{C_j(k,i)}{(z - a_{k,i})^j}, \qquad R_k(z) = \sum_i R_{k,i}(z),$$

$$\Phi_{k,i}(z) = R(z) - R_{k,i}(z), \qquad \Phi_k(z) = R(z) - \sum_{j=k}^{M} R_j(z).$$

We shall prove a number of inequalities for these functions. For convenience of exposition we shall state them in the form of lemmas.

Lemma 1. *The inequality important for the proof of the theorem holds*

$$\iint_{F_{s-1}} |\Phi_s(z)|^2 \, dx \, dy + \sum_{k=s}^{M} \sum_i \iint_{C\Delta_i^{(k)}} |R_{k,i}(z)|^2 \, dx \, dy < AB, \tag{1}$$

where

$$F_{s-1} = C \bigcup_{k=1}^{s-1} \bigcup_i \Delta_i^{(k)}, \qquad F_0 = \Delta, \qquad \Phi_1(z) = P(z), \qquad A = \prod_{k=1}^{\infty} \left(1 + \frac{100 N_{k-1}}{\sqrt{\log n_k}}\right).$$

Lemma 2. *For those z which simultaneously satisfy the inequalities*

$$|z - a_{k,i}| > 2 r'_k \log n_k, \qquad k \geq s+1,$$

the inequality is valid

$$\sum_{k=s+1}^{M}\sum_{i}|R_{k,i}(z)| < \sqrt{2AB}\left\{\sum_{k=s+1}^{M}\frac{2}{\log n_k}\sum_{i}\frac{1}{|z-a_{k,i}|^2}\right\}^{1/2}. \tag{2}$$

In addition to the fact that the function $f(z)$ belongs to $A^2_{E_0}$, we assume its equivalence to zero on some portion E_0: $K \cap E_0$, where K is a disk. We may assume that the functions $R_\varepsilon(z)$ have no poles in this disk. It is not hard to show [2] that on the closed disk $K_0 \subset K$

$$\lim_{\varepsilon\to+0} R_\varepsilon(z) = 0, \qquad (m(E_0 \cap K_0) > 0). \tag{3}$$

We may assume that K_0 is concentric with some square $\delta_{i_0}^{(K_0)}$ and has radius $\dfrac{1}{4N_{K_0}}$.

By $\mathfrak{M}_s$ we denote the set of all those points of the square Δ for which, simultaneously and uniformly with respect to i, the inequalities

$$|z-a_{k,i}|^2 \geq \frac{1}{k^2 N_{k-1}^2}, \qquad k \geq s,$$

are fulfilled.

On the basis of Lemma 2, for $z \in \mathfrak{M}_{s+1}$ we have

$$\sum_{k=s+1}^{M(\varepsilon)}\sum_{i}\left|R_{k,i}^{(\varepsilon)\prime}(z)\right| < \text{const}\cdot\beta_{s+1}; \qquad \beta_{s+1} = \frac{sN_s^2}{\sqrt{\log n_{s+1}}}. \tag{4}$$

Therefore it follows from (3) that, for sufficiently large n, $n \geq n(s)$, on K_0 the inequality

$$\left|\varphi_s^{(\varepsilon_n)}(z)\right| = \left|P_{\varepsilon_n} + \sum_{k=1}^{s}\sum_{i} R_{k,i}^{(\varepsilon_n)}(z)\right| < \text{const}\cdot\beta_{s+1}, \tag{5}$$

is true, where $\varepsilon_n \downarrow 0$ as $n \to \infty$.

Put

$$\gamma_s = \left[\frac{1}{sN_{s-1}}\right] + \mu_s,$$

where $\mu_s \geq 0$ is the minimal number for which the ratio

$$\frac{1}{N_{s-1}} : \gamma_s$$

is an integer. $\Delta_{i,s}^{(k)}$ is the square concentric with $\Delta_i^{(k)}$, with side

$$\frac{1}{N_k} + 2\gamma_{s+1};$$

Δ_s is the square concentric with Δ, with side $1 - 2\gamma_{s+1}$; $K_i^{(k)}$ is the closed disk concentric with $\delta_i^{(k)}$, of radius

$$\frac{1}{4N_k}.$$

We introduce into consideration the domains

$$D_{s,j}^{(m)} = \left\{ \left[\Delta_s \setminus \left(\bigcup_{k=1}^{s} \bigcup_i \Delta_{i,s}^{(k)} \right) \right] + \delta_j^{(m)} \right\} \setminus K_j^{(m)}, \qquad G_{s,j}^{(m)} = D_{s,j}^{(m)} \setminus \left(\bigcup_i \Delta_{i,s}^{(s+1)} \right).$$

and the functions $\omega_{s,j}^{(m)}(z)$, satisfying the conditions

$$\omega_{s,j}^{(m)}(z) = \begin{cases} 1 & \text{on the boundary } K_j^{(m)}, \\ 0 & \text{on the remaining part of the boundary of } D_{s,j}^{(m)}. \end{cases}$$

Let the odd numbers $n_1 < n_2 < \dots < n_s$ have already been chosen in some way. Choose an odd number $n_{s+1} > n_s$ so large that

$$\omega_{s+1,j}^{(m)}(z) > \frac{1}{2}\,\omega_{s,j}^{(m)}(z), \qquad z \in \overline{D_{s,j}^{(m)}} \cap \widetilde{G}_{s,j}^{(m)}, \tag{6}$$

for all $m \le s-1$ and all possible j.

Thus we construct the sequence $\{n_k\}$, at the same time obtaining for it the rate of growth for which Lemma 2 and inequalities (4), (6) are applicable.

Denote

$$D_s = \left\{ \left(\Delta_s \setminus \left(\bigcup_{k=1}^{s} \bigcup_i \Delta_{i,s}^{(k)} \right] \right) \cup \delta_{i_1}^{(k_0)} \right\} \setminus K_0$$

and consider in D_s the harmonic function $\omega_s(z)$, satisfying the boundary condition

$$\omega_s(z) = \begin{cases} 1 & \text{on the boundary } K_0, \\ 0 & \text{on the remaining part of the boundary of } D_s. \end{cases}$$

In the domain D_s one can prove the estimate $\left|\varphi_s^{(s)}(z)\right| < \dfrac{\text{const}}{\alpha_s}$.

With the aid of the generalized maximum principle for subharmonic functions one can prove that on $\overline{D}_s$

$$\log\left|\varphi_s^{(\varepsilon_n)}(z)\right| < 2N_{s-1}^2 \log\frac{\text{const}}{\alpha_s} + \omega_s(z)\log(\text{const}\cdot\beta_{s+1}), \qquad n \geq n(s). \quad (7)$$

Denote

$$G_s = D_s \setminus \left(\bigcup_i \Delta_{i,s}^{(s+1)}\right), \qquad E_s = \bigcap_{k=s}^{\infty}(\overline{D}_k\overline{G}_k).$$

From the last estimate, in view of (4) and (6), it follows that on the closed set $E_r \cap \mathfrak{M}_{r+1}$, for any r, the sequence $\{R_{\varepsilon_n}(z)\}$ tends uniformly to zero. Since $m(E_0 \setminus E_r\mathfrak{M}_{r+1}) \to 0$ as $r \to \infty$, it follows that the function $f(z)$ is equivalent to zero on the set E_0. The theorem is proved.

We note that for $p < 2$ it turns out to be impossible to construct nowhere dense sets of "uniqueness." Moreover, the following theorem is true[3]:

Theorem 3. If $p < 2$, then $A_E^p = \mathcal{L}_p(E)$.

I am pleased to take this opportunity to express my gratitude to S. N. Mergelyan for posing the uniqueness problems and for valuable suggestions in solving them.

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