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Mathematics

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1964

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Abstract

Full Text

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ON THE SOLUTION OF NONLINEAR INTEGRAL EQUATIONS OF THE FIRST KIND

In the present article we shall consider a stable method for solving integral equations of the form

$$K[x, z(s)] = \int_a^b K(x, s, z(s)) ds = u(x), \quad c \leq x \leq d. \quad (1)$$

The problem under consideration is an ill-posed problem, since a solution of this problem does not exist for every $u(x)$, and if for some $\bar{u}(x)$ solutions $\bar{z}(s)$ do exist, then small perturbations of \bar{u} (in the L_2 norm) may correspond to large perturbations of \bar{z} (in the C norm).

In ^(1,2) regularization methods were developed for linear integral equations of the first kind, making it possible, from $\tilde{u}(x)$, the perturbed values of $u(x)$ in the L_2 norm, and δ , the order of the perturbation, to find uniform approximations for $\bar{z}(s) \in W_2^1$. The aim of the present article is to extend these methods to nonlinear equations (see also ^(3,4)). We shall assume that:

- 1°. $K[x, z_1(s)] \neq K[x, z_2(s)]$, if $z_1(s) \neq z_2(s)$.
- 2°. The operator $K[x, z(s)]$ is continuous from C into L_2 , i.e., if a sequence of functions $z_n(s) \in C$ converges uniformly to $z_0(s)$, then $K[x, z_n(s)]_{L_2} \rightarrow K[x, z_0(s)]$ as $n \rightarrow \infty$.
- 3°. $K'_z(x, s, z)$ and $K''_{zz}(x, s, z)$ are continuous in z in a neighborhood of $\bar{z}(s)$.
- 4°. The linear equation of the first kind

$$\int_a^b K'_z(x, s, \bar{z}(s)) w(s) ds = 0$$

has in W_2^1 only the trivial solution.

These conditions, in particular, are satisfied by the nonlinear integral equation of the first kind to which the inverse problem of gravimetry is reduced.

Here we restrict ourselves to the consideration of equations with one variable s . The transition to equations with many variables proceeds as indicated in ⁽¹⁾.

Moreover, similarly to (2), regularization methods are constructed for uniform approximation together with n derivatives, if $\bar{z}(s) \in W_2^{(n+1)}$.

1. Consider the “smoothing” functional

$$M^\alpha[z, \tilde{u}] = N[z(s), \tilde{u}(x)] + \alpha\Omega[z],$$

where

$$N[z(s), \tilde{u}(x)] = \int_c^d |K[x, z(s)] - \tilde{u}(x)|^2 dx$$

quadratic deviation of the operator $K[x, z(s)]$ from the given function $\tilde{u}(x)$, and

$$\Omega[z] = \int_a^b \left[k_1(s) \left(\frac{dz}{ds} \right)^2 + k_0(s) z^2(s) \right] ds, \quad k_0(s), k_1(s) \geq x_0 > 0.$$

Theorem 1. Whatever the function $\tilde{u}(x) \in L_2$ and $\alpha > 0$, there exists at least one function $z^\alpha(s) \in W_2^1$ realizing the minimum of $M^\alpha[z(s), \tilde{u}(x)]$.

Let M_0 be the lower bound of $M^\alpha[z, \tilde{u}(x)]$ for $z(s) \in W_2^1$. This lower bound exists by virtue of the nonnegativity of M^α . Let, further, $z_n(s)$ be a sequence of functions from W_2^1 minimizing M^α , so that

$$M_n = M^\alpha[z_n(s), \tilde{u}(x)] \rightarrow M_0 \quad \text{as } n \rightarrow \infty \quad (M_n \leq M_{n-1}).$$

It is obvious that

$$|z_n(s)| \leq C_1,$$

where C_1 is some constant, since

$$\Omega[z_n] \leq \frac{1}{\alpha} M_1.$$

Thus, from $\{z_n\}$ one can choose a subsequence $z_{n_k}(s)$ converging uniformly to $z_0(s) \in W_2^1$, and $\Omega[z_0] \leq \frac{1}{\alpha} M_1$. Hence it follows that

$$N[z_{n_k}(s), \tilde{u}(x)] \rightarrow N[z_0(s), \tilde{u}(x)] \quad \text{as } k \rightarrow \infty.$$

Thus,

$$M^\alpha[z_0(s), \tilde{u}(x)] = \lim_{k \rightarrow \infty} M_{n_k} = M_0,$$

which proves the existence of a function $z^\alpha(s)$ ($z^\alpha(s) = z_0(s)$) realizing the minimum of $M^\alpha[z(s), \tilde{u}(x)]$.

Theorem 2. Let the function $\bar{u}(x) \in L_2$ correspond to a solution of equation (1), equal to $\bar{z}(s) \in W_2^1$. For any $\varepsilon > 0$ there exists such a $\delta_0(\varepsilon, \bar{z})$ that, if $\|\tilde{u}(x) - \bar{u}(x)\|_{L_2} \leq \delta$ and $q_1\delta^2 \leq \alpha(\delta) \leq q_2\delta^2$ ($q_1 > 0$), then $|z^{\alpha(\delta)}(s) - \bar{z}(s)| \leq \varepsilon$ for $\delta \leq \delta_0$, whatever the function $z^{\alpha(\delta)}(s)$ realizing the minimum of $M^{\alpha(\delta)}[z(s), \tilde{u}(x)]$.

Indeed, since $M^\alpha[z^\alpha(s), \tilde{u}(x)] \leq M^\alpha[\bar{z}(s), \tilde{u}(x)]$, we have: 1) $\alpha(\delta)\Omega[z^{\alpha(\delta)}] \leq \delta^2 + \alpha\Omega[\bar{z}] \leq \alpha\left[\frac{1}{q_1} + \Omega[\bar{z}]\right]$, or $\Omega[z^{\alpha(\delta)}] \leq C_0 = \frac{1}{q_1} + \Omega[\bar{z}]$, i.e., all $z^{\alpha(\delta)}(s)$, for arbitrary δ , belong to the set $Z_0(\bar{z})$, compact in the sense of C , of functions defined by the condition $\Omega[z] \leq C_0$.

- 2) Let U_0 be the set of functions $u(x) = K[x, z]$ ($z \in Z_0$). By the continuity of the mapping $Z_0 \rightarrow U_0$ and the uniqueness of the solution, the inverse mapping $U_0 \rightarrow Z_0$ is defined and continuous, i.e., for any ε there exists such an $\eta_0(\varepsilon, \bar{z})$, depending on $Z_0(\bar{z})$, that if, for $\tilde{u}_1(x), u_2(x) \in U_0$, the norm $\|\tilde{u}_1 - u_2\|_2 \leq \eta_0(\varepsilon, \bar{z})$, then $|z_1(s) - z_2(s)| < \varepsilon$. Further,

$$\begin{aligned} N[z^{\alpha(\delta)}(s), \tilde{u}(x)] &= \int_a^b [u^{\alpha(\delta)} - \tilde{u}(x)]^2 dx = \\ &= \|u^{\alpha(\delta)}(x) - \tilde{u}(x)\|_2^2 \leq \delta^2(1 + q_2\Omega[\bar{z}]), \end{aligned}$$

where $u^{\alpha(\delta)}(x) = K[x, z^{\alpha(\delta)}(s)]$. Thus, since $\bar{z}(s), z^{\alpha(\delta)}(s) \in Z_0(\bar{z})$ and

$$\|u^{\alpha(\delta)}(x) - \bar{u}(x)\|_2 \leq \|u^{\alpha(\delta)} - \tilde{u}\|_2 + \|\tilde{u} - \bar{u}\|_2 \leq \delta C_2$$

$$(C_2 = 1 + \sqrt{1 + q_2\Omega[\bar{z}]}) ,$$

we find that

$$|z^{\alpha(\delta)}(s) - \bar{z}(s)|_C < \varepsilon \quad \text{for } \delta \leq \frac{1}{C_2}\eta_0(\varepsilon, \bar{z}) = \delta_0(\varepsilon, \bar{z}),$$

which proves Theorem 2.

Consider the Euler equation corresponding to the functional $M^\alpha[z(s), \tilde{u}(x)]$. Let $z^\alpha(s)$ be some function from W_2^1 realizing the minimum of $M^\alpha[z, \tilde{u}]$; putting $z = z^\alpha(s) + \eta\zeta(s)$ ($\zeta(s) \in W_2^1$), we obtain

$$\frac{d}{d\eta} [M^\alpha] = \int_c^d F(\xi) \left[\int_a^b K'_z(\xi, s, z^\alpha(s)) \zeta(s) ds \right] d\xi + \alpha \int_a^b [k_1(z^\alpha)'_s \zeta' + k_0 z^\alpha \zeta] ds = 0,$$

where

$$F(\xi) = [K[\xi, z(s)] - \tilde{u}(\xi)]|_{z=z^\alpha(s)} = [u^\alpha(\xi) - \tilde{u}(\xi)].$$

If $k_1 = \text{const}$, then, choosing $\zeta(s)$ to be a piecewise-linear function equal to

$$\zeta(s_0) = \frac{1}{h}, \quad \zeta(s_0 - h) = \zeta(s_0 + h) = 0,$$

and identically equal to zero outside the interval $(s_0 - h, s_0 + h)$, we obtain (for $k_1 = \text{const}$)

$$-\alpha k_1 \frac{z^\alpha(s_0 + h) + z^\alpha(s_0 - h) - 2z^\alpha(s_0)}{h^2} + \left\{ \int_{s_0-h}^{s_0+h} \zeta(s) \left[\int_c^d F(\xi) K'_z(\xi, s, z^\alpha(s)) d\xi + \alpha k_0 z^\alpha \right] ds \right\},$$

and as $h \rightarrow 0$,

$$-\alpha(k_1(z^\alpha)'' - k_0 z^\alpha) + \int_c^d F(\xi) K'_z(\xi, s, z^\alpha(s)) d\xi = 0.$$

It is also not difficult to verify that

$$(z^\alpha)'_{s=a} = (z^\alpha)'_{s=b} = 0.$$

If k_1 depends on s , then analogously

$$-\alpha \left[\frac{d}{ds} \left(k_1 \frac{dz^\alpha}{ds} \right) - k_0 z^\alpha \right] + \int_c^d F(\xi) K'_z(\xi, s, z^\alpha(s)) d\xi = 0, \quad (z^\alpha)'_a = (z^\alpha)'_b = 0.$$

Until now we have left aside the question of the uniqueness of $z^\alpha(s)$.

Theorem 3. Let $\bar{u}(x) \in L_2$ correspond to a function $\bar{z}(s) \in W_2^1$ which is a solution of equation (1). There exists a $\delta_0(\bar{z})$ such that for $\delta < \delta_0(\bar{z})$ there exists only one function $z^{\alpha(\delta)}(s)$ realizing the minimum of the functional $M^{\alpha(\delta)}[z, \tilde{u}(x)]$.

This theorem will be proved if we prove the uniqueness of the solution of the Euler equation with the corresponding boundary conditions. Let

$z_1^{\alpha(\delta)}(s)$ and $z_2^{\alpha(\delta)}(s)$ are two different solutions of the Euler equation and $w(s) = z_1^{\alpha(\delta)} - z_2^{\alpha(\delta)}$. Subtracting the corresponding equations, we find

$$\begin{aligned}
 & -\alpha \left[\frac{d}{ds} \left(k_1 \frac{dw}{ds} \right) - k_0 w \right] + \int_c^d [F_1(\xi) - F_2(\xi)] K'_z(\xi, s, z_1^{\alpha(\delta)}(s)) d\xi + \\
 & + \int_c^d F_2(\xi) [K'_z(\xi, s, z_1^{\alpha(\delta)}(s)) - K'_z(\xi, s, z_2^{\alpha(\delta)}(s))] d\xi = 0, \quad w'(a) = w'(b) = 0,
 \end{aligned}$$

$$F_i(\xi) = [K[\xi, z_i^{\alpha(\delta)}(s)] - \tilde{u}(\xi)] \quad (i = 1, 2).$$

Represent the integral term in the form

$$\begin{aligned}
 & \int_a^b \left\{ \int_c^d K'_z(\xi, s, \bar{z}(s)) K'_z(\xi, s_1, \bar{z}(s_1)) d\xi \right\} w(s_1) ds_1 + \\
 & + \int_a^b \int_c^d \eta(\xi, s, s_1, \delta) w(s_1) ds_1 d\xi + w(s) \eta(s, \delta),
 \end{aligned}$$

where $\eta(\xi, s, s_1, \delta)$ and $\eta(s, \delta)$ are uniformly small as $\delta \rightarrow 0$, which follows from Theorem 2 and the continuity of $K'_z(x, s, \bar{z}(s))$ and $K''_{zz}(x, s, \bar{z}(s))$ in a neighborhood of $\bar{z}(s)$. Multiplying the equation by $w(s)$, integrating and transforming, we obtain

$$\alpha \int_a^b [k_1 (w')^2 + k_0 w^2] ds + \int_c^d \left[\int_a^b K'_z(\xi, s, \bar{z}(s)) w(s) ds \right]^2 d\xi + S = 0,$$

where

$$|S| \leq \eta(\delta) \int_a^b w^2(s) ds$$

and $\eta(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

We shall show that this equality is impossible for sufficiently small $\delta \leq \delta_0$. Indeed,

$$\int_c^d \left[\int_a^b K'_z(\xi, \bar{z}(s)) w(s) ds \right]^2 d\xi \geq \gamma_0 \Omega[w],$$

where γ_0 is the lower bound of the left-hand side of the inequality on the sphere $\Omega[w] = 1$. Here $\gamma_0 > 0$, since

$$\int_a^b K'_z(\xi, s, \bar{z}(s))w(s) ds \neq 0$$

and the sphere $\Omega[w] = 1$ is compact, which proves the impossibility of the last equality for $w(s) \neq 0$ when $\delta < \delta_0$ such that $\eta(\delta) < \gamma_0$.

The numerical algorithms following from the method presented show their high efficiency.

Received
29 III 1964

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