I. L. KANTOR

1964

SovietRxiv

Abstract

Full Text

I. L. KANTOR

CLASSIFICATION OF IRREDUCIBLE TRANSITIVE-DIFFERENTIAL GROUPS

(Presented by Academician I. G. Petrovskii, 8 V 1964)

We give a definition, somewhat strengthened in comparison with (^{1}), of a transitive-differential group of order ν. Let $\mathfrak{G}$ be a local Lie group of transformations acting on a manifold $x^1, x^2, \dots, x^n$, and let G be the corresponding Lie algebra. The elements of the algebra G are operators $f = f^\alpha(x)\partial/\partial x^\alpha$. Each operator f can be represented in the form $f = f_0 + f_1 + f_2 + \cdots$, where $f_k = f_k^\alpha(x)\partial/\partial x^\alpha$, and the $f_k^\alpha(x)$ are homogeneous polynomials of degree k.

We shall say that $\mathfrak{G}$ is a **transitive-differential group of transformations of order** ν if the algebra G has the following properties:

1. The operators $\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^n$ belong to G.
2. Each operator f from G is represented as a finite sum $f_0 + f_1 + \cdots + f_\nu$, and all the operators $f_0, f_1, \dots, f_\nu$ (the homogeneous components of f) also belong to G. For at least one operator f from G, $f_\nu \neq 0$.
3. The algebra G is maximal, i.e. it is not contained in any other Lie algebra of transformations with properties 1 and 2.

Taking into account that the commutation of operators is governed by the law

$$[f, \varphi] = \left(\frac{\partial f^\alpha}{\partial x^\beta}\varphi^\beta - \frac{\partial \varphi^\alpha}{\partial x^\beta} f^\beta\right)\frac{\partial}{\partial x^\alpha},$$

one can define the class of Lie algebras under consideration as follows. Let T_0 denote an n-dimensional vector space, and let T_k be the space of all k-linear symmetric operators acting from T_0 into T_0. If

$$A \in T_k, \qquad B \in T_l, \qquad a \in T_0, \qquad b \in T_0$$

$(k > 0,\ l > 0)$, then set

$$A \circ B = A(B(x, x, \dots, x), x, \dots, x), \qquad A \circ a = A(a, x, \dots, x), \tag{1}$$

$$a \circ A = 0, \qquad a \circ b = 0.$$

Next, in the infinite-dimensional space $T = T_0 + T_1 + T_2 + \cdots$, define the following commutation operation:

$$[A, B] = kA \circ B - lB \circ A. \tag{2}$$

A subspace G of the subspace T, containing T_0, forms the Lie algebra of a transitive-differential group of order ν if: 1) G forms a Lie algebra with respect to the commutation operation (2); 2) $G \subset \sum_{k=0}^{\nu} T_k$, but $G \not\subset \sum_{k=0}^{\nu-1} T_k$; 3) if $A \in G$ and $A = A_0 + A_1 + \cdots + A_\nu$, where $A_k \in T_k$, then all A_k belong to G $(k = 0, 1, \dots, \nu)$; 4) G is maximal, i.e. is not inclu-

is not contained in any other subspace containing T_0 and satisfying conditions 1), 2), 3).

Definition. An algebra G is **irreducible** if there does not exist a subspace $U \subset T_0$ such that, for all operators $A \in G$,

$$A(U, x, \dots, x) \subset U. \tag{3}$$

Otherwise the algebra is called **reducible**.

Theorem 1. *An algebra G of order $v > 2$ is necessarily reducible.*

Each operator of the 2nd order defines a bilinear mapping $T_0 \times T_0$ into T_0 and thereby turns the space T_0 into a certain algebra. We shall say, in a conditional sense, that an operator of the 2nd order is an algebra.

Theorem 2. *The operators of the 2nd order that belong to an algebra G of the 2nd order are Jordan algebras.*

Theorem 3. *An arbitrary Jordan algebra J enters into some algebra G of the 2nd order as a quadratic operator. If the Jordan algebra has a unit, then the algebra G is unique. It consists of the space $J(T_0)$ itself, all linear operators $P_a(x) = ax$ (ax denotes the product of a by x in the sense of J), all differentiations of the algebra J, and all quadratic operators of the form $A(x, x) = ax^2 - 2(ax)x$.*

Theorem 4. *If the Jordan algebra J is simple, then the corresponding algebra G is irreducible.*

Theorem 5. *Let the algebra G be irreducible; then it is simple, and the subalgebra consisting of all linear and quadratic operators from G is maximal nonsemisimple.*

According to Theorem 5, in order to find all irreducible Lie algebras of transitively differential groups, one must go through all simple Lie algebras G and their maximal nonsemisimple subalgebras H. It turns out that not every pair G, H corresponds to the Lie algebra of a transitively differential group. Below the algebras G obtained are listed separately in the complex and in the real case. We note that in the complex case all pairs (G, H) (G simple, H maximal nonsemisimple) were singled out—on the basis of other considerations and not in the form in which they are given below—by V. V. Morozov (^{2}).

Complex case.

I. Series A_n. The algebra G acts on the space T_0 of rectangular complex matrices of type (k, l). Its linear operators have the form

$$X' = AX - XD, \tag{4}$$

where A, D are all possible pairs of square complex matrices of orders, respectively, k and l, with sum of traces equal to zero. The quadratic operators act according to the law

$$X' = XCX, \tag{5}$$

where C ranges over all rectangular complex matrices of type (l, k).

The transitively differential group corresponding to this algebra acts on the space of rectangular matrices of type (k, l) in the following way:

$$X' = (MX + N)(PX + Q)^{-1}, \tag{6}$$

where the matrices M, N are of types (k, k) and (k, l), while P, Q are of types (l, k) and (l, l), and the determinant

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = 1.$$

Note that for $k = n,\ l = 1$ we obtain the projective group in its usual interpretation.

II. Series D_n. The algebra G acts on the space of complex skew-symmetric matrices X of order n. The linear operators have the form

$$X' = AX + XA^T, \tag{7}$$

where A is an arbitrary matrix of order n, and the quadratic operators are

$$X' = XCX, \tag{8}$$

where C is an arbitrary skew-symmetric matrix of order n.

The transitive-differential group acts on skew-symmetric matrices of order n. The transformations of the group have the form (6), where M, N, P, Q are square matrices related by the relations

$$M^TP + P^TM = 0, \qquad N^TQ + Q^TN = 0, \qquad M^TQ + P^TN = E; \tag{9}$$

these relations express the fact that the matrix $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ preserves the form $x_1x_{n+1} + x_2x_{n+2} + \cdots + x_{nx_{2n}}$.

III. Series C_n. The algebra G acts on the space of symmetric matrices X; the linear operators act according to rule (7), and the quadratic ones according to rule (8), only the matrices C in this case are arbitrary symmetric matrices.

The transitive-differential group acts on symmetric matrices of order n. The transformations of the group have the form (6), where M, N, P, Q are square matrices related by the relations

$$M^TP - P^TM = 0, \qquad N^TQ - Q^TN = 0, \qquad M^TQ - P^TN = E; \tag{10}$$

these relations express the fact that the matrix $\begin{pmatrix} M & N \\ P & Q \end{pmatrix}$ preserves the form $(x_1y_{n+1} - y_1x_{n+1}) + \cdots + (x_{ny_{2n}} - y_{nx_{2n}})$.

IV. Series B_n. The algebra G acts on an n-dimensional complex vector space with a given bilinear symmetric nondegenerate form (x, y). The linear operators are: the identity E and all those satisfying the condition $(Ax, y) + (x, Ay) = 0$; the quadratic operators have the form $(x, x)a - 2(a, x)x$, where a is an arbitrary vector.

The transitive-differential group is the conformal group of the complex Euclidean plane.

All the listed transitive-differential groups are obtained from the classical simple Lie groups. As for the exceptional Lie groups, transitive-differential groups arise from them only in the cases E_7, E_6. The generating Jordan algebras are: in the case E_7, the exceptional simple Jordan algebra of Hermitian complex octonion matrices; in the case E_6, a subalgebra of the exceptional simple Jordan algebra.

The real case. The real irreducible Lie algebras of transitive-differential groups are, first, all complex irreducible Lie algebras of transitive-differential groups in which each complex parameter is regarded as two real ones. Second, from real forms of the classical Lie algebras there arise the following. Two algebras that act on rectangular real and on rectangular quaternionic matrices and are constructed analogously to the complex type I; three algebras acting on symmetric real, complex Hermitian, and quaternionic Hermitian matrices, which are constructed analogously to the complex type II. In the last two cases the operation of transposition is replaced by the operation of transposition followed by conjugation. Two algebras acting on real skew-symmetric and on quaternionic co-

Hermitian matrices, which are constructed analogously to the complex case III. In the quaternionic case conjugation is again added to the operation of transposition. Finally, there are real algebras constructed analogously to the complex case IV for an arbitrary nondegenerate symmetric form. We note that in the latter case the transitive-differential groups will be the conformal groups of pseudo-Euclidean spaces.

We shall not investigate here the real algebras G arising from exceptional Lie algebras.

In conclusion, we note that from the list we have given there easily follows a classification of simple Jordan algebras both in the complex and in the real case*.

To obtain it, it suffices to go through all quadratic operators of the irreducible Lie algebras of transitive-differential groups and, from the corresponding Jordan algebras, select only the simple ones.

Ussuriisk State Pedagogical
Institute

Received
25 IV 1964

REFERENCES

1. I. L. Kantor, DAN, **151**, No. 6 (1963).
2. V. V. Morozov, Matem. sborn., **5** (47) (1939).
3. A. A. Albert, Trans. Am. Math. Soc., **59**, No. 3 (1946).

---

* We note that the classification of simple complex Jordan algebras was carried out by A. A. Albert in [3]; as for real algebras, in the works known to us there is no exhaustive list of such algebras, although it is probably not difficult to obtain one by means of the theorems available there.

Note: Figure translations are in progress. See original paper for figures.