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Abstract

Full Text

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ON THE POSSIBLE RATE OF DECREASE OF A SOLUTION OF AN ELLIPTIC EQUATION

(Presented by Academician I. G. Petrovskii, 23 I 1964)

In this note we generalize a theorem of E. M. Landis on the possible rate of decrease of a solution of an elliptic equation.

1. This theorem of Landis was proved in the work ⁽¹⁾ and consists of the following. In the half-strip D_h ($x \geq 0$, $-h < y < h$), $h < 1$, consider the equation

$$A(x, y) \frac{\partial^2 u}{\partial x^2} + 2B(x, y) \frac{\partial^2 u}{\partial x \partial y} + C(x, y) \frac{\partial^2 u}{\partial y^2} + D(x, y) \frac{\partial u}{\partial x} + E(x, y) \frac{\partial u}{\partial y} + F(x, y)u = 0. \quad (1)$$

It is assumed that, for any real ξ and η ,

$$A(x, y)\xi^2 + 2B(x, y)\xi\eta + C(x, y)\eta^2 > \alpha(\xi^2 + \eta^2), \quad \alpha > 0. \quad (2)$$

The coefficients A, B , and C are twice continuously differentiable; D and E are continuously differentiable; A, B , and C themselves, their first and second partial derivatives, and D and E themselves and their first partial derivatives are bounded in modulus by unity; $-1 < F \leq 0$. Under the formulated restrictions, for a solution u satisfying the requirements $|u| < 1$ in D_h , $u(0, y) > a(2^{-8} > a > 0)$, the inequality

$$M(x) > 2^{-2kx/h + \log_2 \log_2^{1/a}} \quad (x > 1),$$

is proved, where

$$M(x) = \sup_{x' \geq x, -h < y < h} |u(x', y)|,$$

and K is a constant depending only on the constant α in inequality (2).

The sharpness of the estimate is shown by the example of the harmonic function $\operatorname{Re} e^{-\pi z/4h}$.

We shall prove that, if one considers the self-adjoint equation

$$\frac{\partial}{\partial x} \left[A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} \right] + \frac{\partial}{\partial y} \left[B(x, y) \frac{\partial u}{\partial x} + C(x, y) \frac{\partial u}{\partial y} \right] = 0, \quad (3)$$

then the theorem formulated above is valid under less restrictive assumptions: it is sufficient to assume that A, B , and C are measurable, bounded in modulus by unity, and satisfy inequality (2). By a solution of equation (3) in a domain D we mean a function $u \in W_2^1$, continuous in D , satisfying the identity

$$\iint_{D'} \left[\left(A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} \right) \frac{\partial \varphi}{\partial x} + \left(B \frac{\partial u}{\partial x} + C \frac{\partial u}{\partial y} \right) \frac{\partial \varphi}{\partial y} \right] dx dy = 0,$$

where D' is an arbitrary domain with smooth boundary, $\overline{D'} \subset D$, and φ is an arbitrary continuous function in $\overline{D'}$ from W_2^1 , $\varphi = 0$ on the boundary of D' .

In what follows the following notation is used everywhere: P_h is the strip $\{-h < y < h\}$, $h < 1$; Γ_1 and Γ_2 are two smooth arcs lying in P_h , each of which has endpoints on both sides of P_h ; Γ_1 and Γ_2 do not intersect, and Γ_2 is situated to the right of Γ_1 ; G is the part of P_h situated between Γ_1 and Γ_2 ; G_i is the part of P_h situated to the right of Γ_i , $i = 1, 2$; X is the abscissa of the rightmost point of Γ_2 ; G_h is the part of P_h situated between Γ_1 and the line $x = X + h$. The letter M (with indices) denotes positive constants depending only on the constant α in inequality (2).

By $\mu_2 G$ we denote the area of the domain G , and by $\mu_1 \gamma$ the length of the curve γ .

2. Let us analyze the author's proof of Landis' theorem. The theorem is obtained as a simple consequence of Lemma 1.

Lemma 1. Let $z > 2^{10}h$ be an arbitrary number; let u be a solution of equation (1) in G_1 , having the following properties: $u > a(2^{-8} > a > 0)$ on Γ_1 , $|u| < 2^{-2z/h + \log_2 \log_2 1/a}$ in G_2 , $|u| < 1$ in G_1 . Then $\mu_2 G > M_1 zh$.

In turn, Lemma 1 is derived from Lemmas 2 and 3.

Lemma 2. Let the domain D lie inside the circle K_R of radius $R < 1$, contain the center O of the circle, and have boundary points on the boundary of the circle; let Γ be that part of the boundary of D which is situated strictly inside K_R . Then for certain M_2 and M_2^* , if $\mu_2 D < M_2^* R^2$, then for any solution u of equation (1), continuous in \overline{D} and satisfying the conditions $u = 0$ on Γ , $u(0) > 0$, the inequality

$$\sup_D u > u(0) e^{M_2 R^2 / \mu_2 D}$$

holds.

Lemma 3. Let u be a solution of equation (1) in some domain $D \supset \overline{G}_h$, satisfying the conditions $u = a > 0$ on Γ_1 , $u = a/2$ on Γ_2 , $u > -a$ in \overline{G}_h . Then $\mu_2 G > M_3 h^2$.

Lemma 2 follows from Lemma 2' and the maximum principle.

Lemma 2'. Let R, O, D and Γ have the same meaning as in Lemma 2. Then for some M'_2 , if $\mu_2 D < M'_2 R^2$, then for any positive solution u of equation (1), continuous in \overline{D} and vanishing at 0 on Γ , the inequality

$$u(0) < \frac{1}{2} \sup_D u$$

holds.

In Landis' work ⁽²⁾, later than ⁽¹⁾, Lemma 2' was proved for the self-adjoint equation (3) under the minimal restrictions on the coefficients formulated above. Thus Lemma 2 also turns out to be valid for equation (3). (In this case, for the self-adjoint equation, the restriction $R < 1$ ceases to be essential—Lemmas 2 and 2' for equation (3) are true for any R .)

Thus, in order to generalize Landis' theorem, it is sufficient to prove Lemma 3 for equation (3). (Similarly to Lemmas 2 and 2', Lemma 3 for equation (3) is true without the assumption $h < 1$, and therefore Lemma 1 and the theorem being proved turn out to be true for any $h > 0$.)

3. We shall reduce Lemma 3 for equation (3), with minimal restrictions on the coefficients, to the following Lemma 3' for equation (3), whose coefficients A, B , and C are continuously differentiable, bounded in absolute value by one, and satisfy inequality (2); we shall denote such an equation by (3').

Lemma 3'. Let u be a solution of equation (3') in some domain $D' \supset \overline{G}_h$, satisfying the conditions $u > \frac{7}{8}a > 0$ on Γ_1 , $u < \frac{5}{8}a$ on Γ_2 , $u > -a$ in \overline{G}_h . Then $\mu_2 G > M_3 h^2$.

Thus, suppose that Lemma 3' for equation (3') has been proved, and let u be a solution of equation (3), satisfying the conditions of Lemma 3.

Take such a domain D' that

$$D \supset \overline{D'} \supset D' \supset \overline{G}_h.$$

Construct a sequence of equations defined in D' :

$$L^{(m)} u^{(m)} \equiv \frac{\partial}{\partial x} \left[A^{(m)}(x, y) \frac{\partial u^{(m)}}{\partial x} + B^{(m)}(x, y) \frac{\partial u^{(m)}}{\partial y} \right] + \frac{\partial}{\partial y} \left[B^{(m)}(x, y) \frac{\partial u^{(m)}}{\partial x} + C^{(m)}(x, y) \frac{\partial u^{(m)}}{\partial x} \right] = 0, \quad m = 1, 2, \dots,$$

such that their coefficients are continuously differentiable and satisfy the conditions

$$|A^{(m)}(x, y)| < 1, \quad |B^{(m)}(x, y)| < 1, \quad |C^{(m)}(x, y)| < 1, \\ A^{(m)} \xi^2 + 2B^{(m)} \xi \eta + C^{(m)} \eta^2 > \alpha(\xi^2 + \eta^2),$$

$$\iint_{D'} [(A - A^{(m)})^2 + (B - B^{(m)})^2 + (C - C^{(m)})^2] dx dy \xrightarrow{m \rightarrow \infty} 0.$$

Let $u^{(m)}$ be a solution of the equation $L^{(m)}u^{(m)} = 0$ in D' , coinciding with u on the boundary D' . The family $\{u^{(m)}\}$, by the maximum principle, is uniformly bounded and, by the Giorgi theorem (3), is equicontinuous in G_h . Therefore there exists a sequence $u^{(m_k)}$, uniformly convergent in G_h to some function u^* . Since $u^{(m)}$ converge in the mean to u , we have $u^* = u$. Choose the index k so large that the function $u^{(m_k)}$ satisfies the conditions $u^{(m_k)} > \frac{7}{8}a$ on Γ_1 , $u^{(m_k)} < \frac{5}{8}a$ on Γ_2 , and $u^{(m_k)} > -a$ in G_h . Since $L^{(m_k)}u^{(m_k)} = 0$ is an equation of the form (3'), according to our assumption, $\mu_2 G > M_3 h^2$.

4. In proving Lemma 3' we use the following theorem of analysis, which is an analogue of the mean-value theorem for functions of two variables. (For functions of two variables this theorem was proved by me; it was then generalized by Landis and me to the case of n variables (4).)

Let, in the ring K in the x, y -plane, situated between two concentric squares with ratio of sides \varkappa , there be defined a twice continuously differentiable function $u(x, y)$ and a quadratic form

$$A(x, y)\xi^2 + 2B(x, y)\xi\eta + C(x, y)\eta^2,$$

satisfying the inequality

$$A(x, y)\xi^2 + 2B(x, y)\xi\eta + C(x, y)\eta^2 > \alpha(\xi^2 + \eta^2), \quad \alpha > 0, \quad (2')$$

the coefficients A, B , and C being continuously differentiable in K and bounded in modulus by unity. Then there is a closed piecewise-smooth curve l , separating the inner boundary of K from the outer one, such that

$$\oint_l \left| \frac{\partial u}{\partial \nu} \right| ds < M \frac{\varkappa}{\varkappa - 1} \text{osc } u,$$

where

$$\frac{\partial u}{\partial \nu} = (A\gamma_1 + B\gamma_2) \frac{\partial u}{\partial x} + (B\gamma_1 + C\gamma_2) \frac{\partial u}{\partial y}$$

(γ_1, γ_2 are the direction cosines of the normal to l); $\text{osc } u = \max_{(x,y) \in K} u(x, y) - \min_{(x,y) \in K} u(x, y)$ is the oscillation of u in K ; M is a constant depending only on the constant α in inequality (2').

5. We proceed to the proof of Lemma 3' for a solution u of equation (3'). Choose $u_1 \geq \frac{7}{8}a$ and $u_2 \leq \frac{5}{8}a$ so that on the level sets $u = u_1$ and $u = u_2$ one has everywhere $\text{grad } u \neq 0$. The level set $u = u_i$ ($i = 1, 2$) consists of a finite or countable number of smooth arcs. It separates Γ_1 from Γ_2 in P_h . Therefore one can find a smooth arc Γ'_i , on which $u = u_i$ ($i = 1, 2$), also separating Γ_1 from Γ_2 in P_h . Denote by G' the part of P_h situated between Γ'_1 and Γ'_2 . It can be shown that there exists a smooth arc Γ ,

having endpoints on both sides of P_h and situated between Γ'_1 and Γ'_2 , such that u is constant on Γ , with $u_1 > u_\Gamma > u_2$, and for any arc $\gamma \subseteq \Gamma$

$$\int_\gamma \frac{\partial u}{\partial \nu} ds > \alpha \frac{(u_1 - u_2)(\mu_1 \Upsilon)^2}{\mu_2 G'}.$$

Consider the point \mathcal{P} of intersection of Γ with the straight line $y = 0$, and construct two concentric squares K_h and K_{2h} with center at \mathcal{P} and with sides h and $2h$. Let K be the closed ring enclosed between these squares. Let K_1 be the component, containing the point \mathcal{P} , of the subset of K_{2h} on which u is defined and satisfies the inequalities $u_1 > u > -a$. Every boundary point of K_1 belongs either to the boundary of K_{2h} , or to the level set $u = u_1$; moreover, those boundary points of K_1 at which $u = u_1$ (and, consequently, $\text{grad } u \neq 0$) lie on no more than a countable set of smooth arcs $\gamma^{(i)}$ ($i = 1, 2, \dots$). From the set K_1 extend u in a twice continuously differentiable manner to the ring K , preserving the inequality $\text{osc } u < 2a$. Also extend to the ring K the coefficients A, B , and C of equation (3') in a continuously differentiable manner, preserving the inequalities $|A| < 1, |B| < 1, |C| < 1$ and

$$A\xi^2 + 2B\xi\eta + C\eta^2 > \alpha(\xi^2 + \eta^2)$$

(the extensions of the functions u, A, B , and C are denoted by the same letters u, A, B , and C).

According to item 4, there exists a closed piecewise-smooth curve l , separating the inner boundary K from the outer one and such that $\oint_l \left| \frac{\partial u}{\partial \nu} \right| ds < 4Ma$.

The curve l is intersected, obviously, by only a finite number of arcs $\gamma^{(i)}$. The rightmost point of l lies to the left of the line $x = X + h$. Therefore on l one can find a finite number of arcs $l^{(i)}$, which, together with the arcs $\gamma^{(i)}$, bound a domain B containing the point P^0 and lying, together with its boundary, inside D' .

Denote by γ the smallest arc belonging to the arc Γ and having its endpoints on the curve l . The arc γ divides the domain B into two domains. Denote by B_0 that one of them for which the outward normal to the boundary β_0 coincides on γ , in direction, with $\text{grad } u$. Denote $\beta_0 \cap l$ by l_0 , and $\beta_0 \cap (\bigcup_i \gamma^{(i)})$ by γ_0 . Since on γ_0 the outward normal coincides in direction with $\text{grad } u$, we have $\int_{\gamma_0} \frac{\partial u}{\partial \nu} ds > 0$; since l_0 is part of l , we have $\int_{l_0} \left| \frac{\partial u}{\partial \nu} \right| ds < 4Ma$; finally, since $\mu_1 \gamma \geq h, u_1 - u_2 \geq a/4$, and $\mu_2 G' < \mu_2 G$, we have $\int_\gamma \frac{\partial u}{\partial \nu} ds > \frac{\alpha a h^2}{4\mu_2 G}$.

Since $B_0 \subset D'$, the equality holds

$$\iint_{B_0} \left\{ \frac{\partial}{\partial x} \left[A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} \right] + \frac{\partial}{\partial y} \left[B(x, y) \frac{\partial u}{\partial x} + C(x, y) \frac{\partial u}{\partial y} \right] \right\} dx dy = 0.$$

Applying Green's formula to its left-hand side, we obtain $\int_{\beta_0} \frac{\partial u}{\partial \nu} ds = 0$.

Thus,

$$\int_{\gamma \cup \gamma_0} \frac{\partial u}{\partial \nu} ds = \left| \int_{I_0} \frac{\partial u}{\partial \nu} ds \right| \leq \int_{I_0} \left| \frac{\partial u}{\partial \nu} \right| ds,$$

or $\alpha h^2 / 4\mu_2 G < 4M$. Denoting $\alpha/16M$ by M_3 , we obtain $\mu_2 G > M_3 h^2$, as was required to prove.

6. Up to now we have considered the homogeneous self-adjoint equation (3). The theorem is also valid for the equation

$$\frac{\partial}{\partial x} \left[A(x, y) \frac{\partial u}{\partial x} + B(x, y) \frac{\partial u}{\partial y} \right] + \frac{\partial}{\partial y} \left[B(x, y) \frac{\partial u}{\partial x} + C(x, y) \frac{\partial u}{\partial y} \right] + D(x, y) \frac{\partial u}{\partial x} + E(x, y) \frac{\partial u}{\partial y} + F(x, y)u = 0,$$

the coefficients being measurable and, in modulus, less than one. If, moreover, it is assumed that $F \leq 0$, then Lemmas 2 and 3 remain true and their proof changes only insignificantly. If nothing is assumed about the sign of F , then the theorem is true for sufficiently small h .

In conclusion I express my deep gratitude to E. M. Landis for posing the problem and for help in the work.

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Note: Figure translations are in progress. See original paper for figures.

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