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Abstract

Full Text

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THE GLOBAL CAUCHY PROBLEM FOR NONLINEAR EQUATIONS AND SOME QUASILINEAR SYSTEMS OF FIRST ORDER WITH MANY VARIABLES

(Presented by Academician L. S. Pontryagin on 24 XII 1963)

The paper considers the global Cauchy problem for the equation

$$u_t + f(t, x, u, u_x) = 0, \tag{1}$$

$$x = (x_1, \dots, x_n), \qquad u_x = \operatorname{grad}_x u, \qquad u_t = \frac{\partial u}{\partial t}$$

with the initial condition

$$u(0, x) = u_0(x) \tag{2}$$

(for simplicity of formulation we confine ourselves here to the case when the function f depends only on u_x). We assume that the function f is twice continuously differentiable, $f(0) = 0$, and for any real $\xi_1, \dots, \xi_n$,

$$\sum_{i=1}^{n} \xi_i^2 = 1,$$

$$0 < \lambda(u_x) \le \sum_{i,j=1}^{n} f_{ij}(u_x)\, \xi_i \xi_j, \qquad f_i = \frac{\partial f}{\partial u_{x_i}}, \qquad f_{ij} = \frac{\partial^2 f}{\partial u_{x_i} \partial u_{x_j}}, \tag{3}$$

where the function $\lambda(u_x)$ is continuous; the function $u_0(x)$ is bounded ($|u_0| \le M_0$) and satisfies a Lipschitz condition in the totality of its arguments in the whole space, with constant $K_0 \ge 0$.

In the present paper a definition is given of a generalized solution of problem (1)–(3) in the half-space $\{t \ge 0\}$; its existence and uniqueness are proved, and the behavior of solutions as $t \to +\infty$ is also considered. We obtain the solution of problem (1)–(3) as the limit, as $\varepsilon \to 0$, of solutions $\{u^\varepsilon(t, x)\}$ of the Cauchy problem for the equations

$$u_t + f(u_x) = \varepsilon \Delta u \tag{4}$$

with the initial condition (2); in connection with this, the Cauchy problem for a quasilinear parabolic equation is studied.

From the results concerning problem (1)–(3), as consequences there follow existence and uniqueness theorems for the generalized solution of the global Cauchy problem for the quasilinear system

$$\frac{\partial v_i}{\partial t} + \frac{\partial}{\partial x_i} f(v_1, \dots, v_n) = 0, \qquad i = 1, \dots, n, \tag{5}$$

with the initial condition

$$v_i(0, x) = v_i^0(x) \in L_\infty(E_n(x)), \tag{6}$$

where the vector $v^0 = (v_1^0, \dots, v_n^0)$ satisfies the condition $\operatorname{rot} v^0 = 0$ in the weak sense: $\int (\omega_{x_i} v_j^0 - \omega_{x_j} v_i^0)\, dx = 0$ for any smooth finite function $\omega(x)$. We note that the properties of the solutions of problem (5), (6) are, in a certain sense, analogous to the properties of solutions of the equation $u_t + (\varphi(u))_x = 0$ with convex function $\varphi(u)$, studied in work [1].

In the case of one spatial variable, problem (1)–(3) was studied in ([2–4]); some questions connected with problem (1)–(3) were considered in ([5,6]).

1. We shall say that a function $u(t, x)$, defined in the half-space $\{t \geqslant 0\}$, **belongs to the class** $\mathcal{L}(K)$ if $|u| \leqslant K$ and

$$|u(t + \Delta t, x + \Delta x) - u(t, x)| \leqslant K(|\Delta t| + |\Delta x|), \qquad \Delta t \geqslant 0.$$

Definition 1. A function $u(t, x) \in \mathcal{L}(K)$ (for some K) is called a **generalized solution** of problem (1)–(3) in the half-space $\{t \geqslant 0\}$ if $u(t, x)$ satisfies equation (1) almost everywhere in $\{t \geqslant 0\}$, $u(0, x) = u_0(x)$, and for any vector $l = (l_1, \dots, l_n)$

$$\frac{\Delta_l^2 u}{|l|^2} = \frac{u(t, x + l) - 2u(t, x) + u(t, x - l)}{|l|^2} \leqslant \frac{C}{t}, \qquad C = \text{const.} \tag{7}$$

Theorem 1. *The generalized solution of problem (1)–(3) is unique.*

Suppose that there exist two solutions $u^1(t, x)$ and $u^2(t, x)$. Denote by $\{u_m^k(t, x)\}$, $k = 1, 2$, sequences of functions belonging to $\mathcal{L}(K) \cap C^2$, converging uniformly as $m \to \infty$ to $u^k(t, x)$ in any compact domain $Q \subset \{t \geqslant 0\}$, with

$$\|\operatorname{grad} u_m^k - \operatorname{grad} u^k\|_{L_1(Q)} \to 0$$

as $m \to \infty$, and, in any direction l in $E_n(x)$,

$$\partial^2 u_m^k / \partial l^2 \leqslant C/t$$

(as u_m^k one may take the corresponding averaged functions). Let $\omega_m = u_m^1 - u_m^2$. It is not difficult to see that, for any smooth function $g(t,x)$ with compact support in the layer $\{0 < \delta \leqslant t \leqslant T\}$, equal to zero for $t = T$, the integral identity

$$\int_\delta^T \int \omega_m \left[g_t + \sum_{i=1}^n \int_0^1 f_i(\dots)\, d\tau\, g_{x_i} + \sum_{i,j=1}^n \int_0^1 f_{ij}(\dots) \frac{\partial^2}{\partial x_i \partial x_j} (\tau u_m^1 + (1-\tau) u_m^2)\, d\tau\, g \right] dx\, dt + \int g\omega_m\big|_{t=\delta}\, dx =$$

$$= \int_\delta^T \int \alpha_m g\, dx\, dt,$$

where

$$(\dots) = (\tau (u_m^1)_{x_1} + (1-\tau)(u_m^2)_{x_1}, \dots, \tau (u_m^1)_{x_n} + (1-\tau)(u_m^2)_{x_n})$$

and

$$\|\alpha_m(t,x)\|_{L_1(Q)} \to 0$$

as $m \to \infty$. Denote by A the differential operator with respect to g in the square brackets, and as $g(t,x)$ in the integral identity take the solution of the Cauchy problem

$$Ag = \Phi(t,x), \qquad g(T,x) = 0,$$

where $\Phi(t,x) \leqslant 0$ is a smooth function equal to zero for $|x| \geqslant N$ and $\delta_0 \geqslant t \geqslant 0$. Clearly, $g(t,x) = 0$ outside some fixed cylinder $Q\{\Omega \times [0,T]\}$. Since, by (3), the matrix

$$\|a_{ij}\| = \|f_{ij}(\dots)\|$$

is positive definite, for any point (t_0, x_0) one can specify an orthogonal change of coordinates $x \to y$ such that

$$H = \sum_{i,j=1}^n a_{ij} (u_m^k)_{x_i x_j}\big|_{(t_0,x_0)} = \sum_{i=1}^n \alpha_i (u_m^k)_{y_i y_i}, \qquad 0 < \alpha_i < M_1,$$

whence

$$H \leqslant n M_1 C / t_0,$$

and therefore $|g(t,x)| \leqslant C_\delta$ for $t \geqslant \delta > 0$. Moreover, one can show that

$$\int |g(t,x)|\, dx \leqslant M_2$$

for all m and $t \in [0,T]$. Fix a number $\varepsilon > 0$; choose first δ sufficiently small, and then m_0 so large that for $m \geqslant m_0$

$$\left| \int_0^T \int \omega_m \Phi(t,x)\, dx\, dt \right| \leqslant \varepsilon.$$

Letting $m \to \infty$ and taking into account the arbitrariness of ε and of the function $\Phi(t,x) \leqslant 0$, we conclude that $u_1 \equiv u_2$.

Theorem 2. The generalized solution of problem (1)–(3) exists.
For the proof, consider the family $\{u^\varepsilon(t,x)\}$ of solutions of the Cauchy problem for equation (4) with initial condition $u^\varepsilon(0,x) = u_0^\varepsilon(x)$, where $u_0^\varepsilon \to u_0$ as $\varepsilon \to 0$, $u_0^\varepsilon \in \mathcal{L}(K_0)$, $|u^\varepsilon_{0x_ix_i}| \le 1/\varepsilon$ (we first establish an existence theorem for the solution of the Cauchy problem for a quasilinear parabolic equation, which in the case of equation (4) is valid only under the smoothness condition on the function f). By means of the maximum principle we obtain the estimates

$$|u^\varepsilon| \le M_0, \qquad |u^\varepsilon_x| \le K, \qquad |u^\varepsilon_t| \le K.$$

We prove the compactness of $\{u^\varepsilon_{x_i}\}$ in the norm L_1 on any cylinder $Q^\delta\{\Omega\times[\delta,T]\}$, $\delta > 0$. To simplify the proof, assume that the functions $u_0^\varepsilon(x)$ are finite. Then $(u^\varepsilon_{x_ix_i})^+ \to 0$ as $|x| \to \infty$ uniformly for $0 \le t \le T$.

Lemma. Let $u^\varepsilon(t,x)$ be a solution of equation (4); then for any direction l in the space $E_n(x)$

$$\frac{\partial^2 u^\varepsilon}{\partial l^2} \le \frac{C}{t}, \qquad 2\delta e\,\frac{1}{C} = \min_{|u_x|\le K} \lambda(u_x). \tag{8}$$

Obviously, it suffices to prove the estimate $u^\varepsilon_{x_1x_1} \le (C/t)$, regarding Δ in equation (4) as some elliptic operator with constant coefficients. Let $w = tu^\varepsilon_{x_1x_1}$; differentiating equation (4), we find that

$$0 = \frac{\partial w}{\partial t} - \frac{w}{t} + t\sum_{i,j=1}^n f_{ij}\frac{\partial u_{x_1}}{\partial x_i}\frac{\partial u_{x_1}}{\partial x_j} + \sum_{i=1}^n f_i\frac{\partial w}{\partial x_i} - \varepsilon\Delta w \ge$$

$$\ge w_t + \frac{\lambda(u_x)w^2 - w}{t} + \sum_{i=1}^n f_i\frac{\partial w}{\partial x_i} - \varepsilon\Delta w.$$

At a point of positive maximum of the function w the inequality $\lambda(u_x)w^2 - w \le 0$ holds; hence $w = tu^\varepsilon_{x_1x_1} \le C$.

From estimate (8) it follows that

$$\|u^\varepsilon_{x_ix_i}\|_{L_1(\Omega)} \le 2\left(\frac{C}{t}d^n + Kd^{n-1}\right),$$

where d is the diameter of the domain Ω. Since for $i \ne j$ the operators

$$L_\pm u \equiv u_{x_ix_i} \pm u_{x_ix_j} + u_{x_jx_j} = u_{y_iy_i} + u_{y_jy_j}$$

under a suitable change of variables

$$x_i = \beta_i y_i + \gamma_i y_j, \qquad x_j = \beta_j y_i + \gamma_j y_j,$$

we have

$$L_\pm u^\varepsilon \le (C_1/t)$$

and

$$|u^\varepsilon_{x_i x_j}| \le (C_1/t) - u^\varepsilon_{x_i x_i} - u^\varepsilon_{x_j x_j}.$$

By means of the estimates obtained, we establish that the sets $\{u^\varepsilon_{x_i}\}$ are compact in $L_1(Q^\delta)$. By a diagonal process we select a sequence $u^{\varepsilon_m}(t,x)$ converging as $\varepsilon_m \to 0$ to $u(t,x)$ uniformly in any cylinder $Q\{\Omega \times [0,T]\}$, and moreover

$$u^{\varepsilon_m}_{x_i} \to u_{x_i} \quad \text{in } L_1(Q^\delta)$$

for any $\delta > 0$. From equation (4),

$$u^{\varepsilon_m}_t \to u_t \quad \text{in } L_1(Q^\delta).$$

It is not difficult to verify that $u(t,x)$ is a generalized solution of problem (1)–(3)*.

Theorem 3. Let $u(t,x)$ be a generalized solution of problem (1)–(3) in $\{t \ge 0\}$, and let

$$f(0) = f_i(0) = 0, \qquad i = 1, \dots, n.$$

Then

$$u(t,x) \to \mu = \inf u_0(x) \quad \text{as } t \to +\infty$$

uniformly in x in any compact domain $\Omega \subset E_n(x)$.

The proof is analogous to the proof of Theorem 3 in (3).

2. Definition 2. A generalized solution of problem (5), (6) in the half-space $\{t \ge 0\}$ is a bounded measurable vector function

$$v(t,x) = (v_1, \dots, v_n)$$

such that, for any smooth functions

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* Taking into account the uniqueness of the generalized solution (Theorem 1), we conclude that

$$u^\varepsilon(t,x) \to u(t,x)$$

as ε tends arbitrarily to 0.

$g^i(t,x)$, finite with respect to x_i and equal to zero for $t \geqslant T$, the integral identities

$$\iint_{\{t\geqslant 0\}} \left[g^i_t v_i + g^i_{x_i} f(v_1, \dots, v_n)\right] dx\,dt + \int g^i(0,x) v^0_i(x)\,dx = 0 \qquad (9)$$

$$i = 1, \dots, n,$$

are satisfied, and for any smooth finite function $\omega(x) \geqslant 0$ and nonnegative constant matrix $\|a_{ij}\| = a$, $|a_{ij}| < 1$,

$$\int \left[(\omega_x, av) + \frac{C}{t}\,\omega\right] dx \geqslant 0, \qquad C = \text{const.} \tag{10}$$

Theorem 4. *The generalized solution of problem (5), (6) exists and is unique. This solution is the limit, as $\varepsilon \to 0$, of the solutions of the Cauchy problem for the quasilinear parabolic system obtained from system (5) if the term $\varepsilon \Delta v_i$ is added to the right-hand side of its i-th equation.*

Remark. Just as in the case $n = 1$, one can study problem (1)–(3) in the case where $u_0(x)$ is an arbitrary bounded lower semicontinuous function (see ([2])).

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Note: Figure translations are in progress. See original paper for figures.